A3
Relativity

The Theory of

GENERAL RELATIVITY

and

GRAVITATION

Based on a course of lectures delivered at the Conference on Recent Advances in Physics held at the University of Toronto, in January, 1921

BY

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PREFACE

At the Conference on Recent Advances in Physics held in the Physics Laboratory of the University of Toronto from January 5 to 26, 1921, a course on Einstein's Relativity and Gravitation Theory, consisting of fifteen lectures and two colloquia, was delivered by the author. The first six of these lectures were devoted to what is known as Special Relativity, and the remaining ones to Einstein's General Relativity and Gravitation Theory and to relativistic Electromagnetism. In view of the time limitations only the essentials of these theories were dealt with, due attention, however, being given to the critically conceptual side of the subject. The University was kind enough to undertake the publication of that part of the course which dealt with general relativistic questions, on the express understanding that my prospective readers should be assumed to be already familiar with the special theory of relativity. In this connection it was suggested by Prof. McLennan that those unacquainted with the older theory should be referred to my book of 1914 (The Theory of Relativity, Macmillan, London) and that it would therefore be desirable to make the present volume, as much as possible, uniform in exposition and style with that work. With such requirements in view this little book was shaped, only a few pages at the beginning having been used in recalling the essentials of the special relativity theory.

The treatment, as compared with the Toronto lectures, has been made somewhat more systematic and the subject matter has, here and there, been considerably extended. In this respect the author has been partly influenced by a larger course on Relativity, Gravitation and Electromagnetism delivered, in the time of writing, during the last Summer Quarter at the University of Chicago. Such is especially the case with Chapter III in which care has been taken to give the readers a systematic exposition of the calculus of generally covariant beings called Tensors. The exposition follows here mainly upon Einstein's own presentation of the subject, with the difference, however, that due emphasis has been laid upon the distinction between metrical and non-metrical properties of tensors. But even in this chapter

technicalities have been avoided, stress being laid upon the guiding principles of this new, or rather newly revived, and most powerful mathematical method. It seems hard to say whether Einstein's admirable theory has or has not a long life before it in the domain of Physics proper. But independently of its fate the time applied for studying the Tensor Calculus and acquiring some skill in handling it will be well spent.

The plan of the remainder of the book will be sufficiently clear from the titles of the chapters and sections arrayed in the table of Contents. Such matter as seemed for the present too speculative and controversial has been relegated to the Appendix where, however, also some points concerning the curvature properties of a manifold have found their place, not only as a preparatory to Einstein's cosmological speculations but perhaps as a useful supplement to Chapter III.

The book is felt to be far from being complete. But as it is, it is hoped that it will give the reader a good insight into the guiding spirit of Einstein's general relativity and gravitation theory and enable him to follow without serious difficulties the deeper investigations and the more special and extended developments given in the large and rapidly growing number of papers on the subject.

Some of my readers will miss, perhaps, in this volume the enthusiastic tone which usually permeates the books and pamphlets that have been written on the subject (with a notable exception of Einstein's own writings). Yet the author is the last man to be blind to the admirable boldness and the severe architectonic beauty of Einstein's theory. But it has seemed that beauties of such a kind are rather enhanced than obscured by the adoption of a sober tone and an apparently cold form of presentation.

My thanks are due to Sir Robert Falconer and to Prof. J. C. McLennan for promoting the cause of this publication, to Prof. R. A. Millikan and Prof. Henry G. Gale of the University of Chicago for reading part of the proofs, and to the University of Toronto Press for the care bestowed on my work.

L. S.

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CHAPTER I.

Special Relativity recalled. Foundations of General Relativity and Gravitation Theory.

In accordance with the purpose and the origin of this volume* its readers are assumed to have already made themselves familiar with the essentials of Einstein's older or special Relativity. It will be enough, therefore, to recall here very concisely what of that theory may be conducive to, and even necessary for, a thorough grasping of the structure and the aims of the more general theory, and of the spirit pervading it.

1. First of all, then, out of all thinkable *reference*-frameworks, the special relativity is concerned only with a certain privileged class of frameworks or systems of reference, *the inertial* systems. Of these there are ∞^3. If S, say the 'fixed-stars' system, is one of them, any other rigid system S' of coördinate axes moving relatively to S with any uniform and purely translational velocity v, in any direction whatever, is again an inertial system or belongs to the same privileged class. And the systems thus derived from S, or from one another,** exhaust the class. Since the size or absolute value of the relative velocity implies one scalar datum, and its direction two more such data, all independent of one another, there is just a triple infinity of inertial systems,† as already stated. Not that the special relativity theory abstains from considering accelerated, *i.e.* non-uniform motion of particles within any of these systems; but it does not contemplate any frameworks other than the inertial ones as systems of refer-

*Cf. Preface.

**If S' moves uniformly with respect to S, and S'' with respect to S', so does S'' with respect to S. If the reader so desires, he may consider this as a postulate.

†The purely spatial orientation of the axes, implying further free data, is irrelevant in the present connection.

ence, and cannot, nor does it propose to deal with them. It is unable, for instance, to transform the course of phenomena from the S system to the spinning Earth or to an accelerated carriage as reference systems.

2. Keeping this in mind, the first main assumption of the older theory, known as the Special Relativity Principle, can be briefly stated by saying that it requires the laws of physical phenomena to be *the same* whether they are referred to one or to any other inertial system. In short, the maxim of the 1905—Relativity was: Equal laws for all inertial systems.

The italicized words are, mathematically speaking, at first somewhat vague. In fact, they are intended to stand for 'the same form of mathematical equations expressing the laws.' Now, since this implies the use of some magnitudes, such as the coördinates and the time, or the electric and the magnetic vectors (forces), in each of the said systems, the requirement of mathematical 'sameness' remains cloudy until we are told what dictionary is to be used to translate the language of one into that of any other inertial system, or technically, to transform from the non-dashed to the dashed variables. This vagueness, however, soon disappears, giving place to precision, in the next fundamental step of the theory as will be seen presently.

The attentive reader might here object by saying that 'sameness of laws' means absence of difference, absence of observable different behaviour (of moving bodies or of electric waves) in passing from an S to an S', and that, therefore, to begin with, no mathematical magnitudes or equations are required. But actually we are, perhaps forever, confined to one (approximately) inertial system, our planet, and are thus unable to observe *directly* the permanence of behaviour in passing to another system of reference. The only way open to us is to proceed, through more or less long chains of abstract reasoning, from the principle of relativity to some observable prediction, and such processes are scarcely practicable without the use of mathematical symbols and equations.

3. The second assumption, called the Principle of Constant Light-velocity, apart from its own importance, provides for the need just explained, its true office in the structure of the theory being to set an example of a 'physical law' which is postulated to satisfy rigorously the first assumption. It

runs thus: Light is propagated, *in vacuo*, relatively to any inertial system, with a velocity c, constant and equal for all directions, no matter whether the source emitting it is fixed or moving with respect to that system. This is shortly referred to as *uniform and isotropic* light propagation in any inertial system. The light velocity, in empty space, plays the part of a universal constant,—which rôle, however, it will readily give up in generalized relativity.

The reader is well acquainted with the mathematical expression of the consequence of these two assumptions (together with a tacit requirement of *formal* equivalence of any two inertial systems S, S'), to wit, *the invariance* of the quadratic form

$$c^2t^2-x^2-y^2-z^2, \tag{a}$$

where x, y, z are the cartesian co-ordinates and t the time of the S-system. That is to say, if x', y', z', t' be the cartesian co-ordinates and the time used in any other inertial system S', (a) should transform into

$$c^2t'^2-x'^2-y'^2-z'^2. \tag{a'}$$

As a matter of fact, what was originally required was that the equation $(a)=0$ should transform into $(a')=0$, and this would be satisfied by putting $(a')=\lambda\ .\ (a)$, where λ is independent of x, y, z, t but might be some function of v, the relative velocity of S, S'. This, however, would amount to a distinction between the two systems, at least a formal one, unless $\lambda=1$. If, therefore, equal rights are claimed not only physically but also *formally*, mathematically, for all inertial systems, we have $(a)=(a')$, that is to say, the quadratic form (a) is raised to the dignity of an *invariant*.

There is, certainly, nothing to object to in such a procedure, especially as it carries simplicity with itself. Yet these remarks did not seem superfluous, especially as there is among the relativists a strong tendency to a certain kind of hypostasy of the said quadatic form* (by declaring it to be more

*Intensified more recently in the case of the more general (differential) quadratic form playing a fundamental rôle in the newer relativity theory, as will be seen hereafter.

'objective, real or intrinsic' than space-distance or time) just because it "is" invariant,—and forgetting that we have deliberately made it invariant.

4. Meanwhile, returning to the quadratic expression (a), let us write it down for a pair of events infinitesimally near to one another in space and time. Thus, writing x_1, x_2, x_3, x_4 for x, y, z, ct, the statements made above can be expressed by saying that the quadratic differential form

$$ds^2 = dx_4^2 - dx_1^2 - dx_2^2 - dx_3^2 \tag{1}$$

should be *invariant* with respect to the passage from one inertial system S to any other such system S'. The differential form is here preferable to the original one, as it will be helpful in paving the way for general relativity.

As is well known, this requirement of invariance gives the rule of transformation of the variables x_ι into those x'_ι of the S'-system, called *the Lorentz transformation.* If both the x_1 and the x'_1 axes are taken along the line of motion of S' relatively to S, with the velocity $v = \beta c$, if further the x_2, x_3—axes are taken parallel to those of x'_2, x'_3, and if the convention $x'_1 = x'_4 = 0$ for $x_1 = x_4 = 0$ is adopted, the Lorentz transformation assumes the familiar form

$$x'_1 = \gamma(x_1 - \beta x_4),\ \ x'_2 = x_2,\ \ x'_3 = x_3,\ \ x'_4 = \gamma(x_4 - \beta x_1) \tag{2}$$

where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$. *Vice versa*, we have, by solving (2),

$$x_1 = \gamma(x'_1 + \beta x'_4),\ x_2 = x'_2,\ x_3 = x'_3,\ x_4 = \gamma(x'_4 + \beta x'_1),$$

showing the complete (including the formal) equivalence of the two systems. Let us keep well in mind, for what is to follow, that this transformation is a linear one, with constant coefficients, and that special relativity, concerned with inertial systems only, does not contemplate any other space-time transformations.

Every tetrad of magnitudes X_ι ($\iota = 1$ to 4) which are transformed as the x_ι, is called a *four-vector* or, after Minkowski, a world-vector of the first kind. Such four-vectors are, in addition to dx_ι or x_ι itself, their prototype, the four-velocity dx_ι/ds and the four-acceleration of a moving particle, the electric four-current, and so on. To every vector X_ι

belongs a scalar or invariant $X_4^2-X_1^2-X_2^2-X_3^2$, its only invariant with respect to the Lorentz transformation. But we need not stop here to reconsider the properties of the four-vector and other world-vectors, such as the six-vector, which constituted the only lawful material of the older relativity for writing down laws of Nature,—especially as we shall soon return to these mathematical entities as particular cases of *tensors* of various ranks which are indispensable to the general theory of relativity.

On the other hand we may profitably dwell yet a while upon the quadratic form (1) itself, the square of *the line-element*, as ds is called. Granted the assumptions of special relativity, this expression becomes the fundamental quadratic differential form of the four-manifold, the world or space-time, in exactly the same way as

$$d\sigma^2=dx^2+dy^2$$

is the fundamental form of a flat two-space or surface, and more generally,

$$d\sigma^2=Edu^2+2Fdudv+Gdv^2$$

that of any surface, and

$$d\sigma^2=dr^2+R^2\sin^2\frac{r}{R}\,(\sin^2\phi\;\;d\theta^2+d\phi^2)$$

the fundamental differential form of any three-space of constant curvature R^{-2}.† Now, it is enough to open any book on differential geometry to see that, with the usual assumptions of continuity, etc., the whole geometry, *i.e.*, all metrical properties of the two-space or the three-space in question are completely determined by the corresponding differential forms. Their geodesics or, within restricted regions at least, their shortest lines, the angle relations, and their whole trigometry, all this is fully determined provided the coefficients of the differentials, such as E, etc., appearing in

†According as R^2 is positive, zero or negative, we have an elliptic, euclidean (or parabolic) or hyperbolic space. In the latter case $R\sin\frac{r}{R}$ becomes $\bar{R}\sinh(r/\bar{R})$, where $\bar{R}^2=-R^2$.

the fundamental form are given functions of the variables.*

This deterministic mastery of the quadratic differential form has been, as far back as 1860, technically extended to spaces or manifolds of four and, in fact, of any number of dimensions,—although, not being sufficiently sensational, it never attracted the attention of anybody beyond a few specialists.

In much the same way all the metrical properties of the four-dimensional world of the special relativist should be, and are, derivable from the fundamental form (1) belonging, or rather allotted to it. This is, from the point of view of the poly-dimensional differential geometer, but a very special, in fact, the most simple quadratic form in four variables. For it contains but the squares of their differentials, and the coefficients of these are all constant, which—in view of the sequel—it may be well to bring into evidence by writing (1)

$$ds^2 = g_{\iota\kappa} dx_\iota dx_\kappa, \tag{1a}$$

to be summed over ι, $\kappa = 1, 2, 3, 4$, tabulating the coefficients, thus

$$\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \tag{1b}$$

and calling this array of special coefficients the inertial or *the galilean* $g_{\iota\kappa}$. We shall denote them in the sequel by $\overline{g_{\iota\kappa}}$. To give this array is as much as to give the form (1a), and herewith the properties of the world,—for it is manifestly irrelevant how we call or denote the four corresponding variables. The values of the $g_{\iota\kappa}$ being given, the properties

*To be rigorous we should have said 'all properties of a restricted region of the contemplated manifold'; for certain properties of the manifold *as a whole* are still left free. The choice, however, is limited to a small number of discrete possibilities. Thus, for example, there are two kinds of elliptic space, the *spherical* or *antipodal*, and the *polar* or *elliptic* proper. In the former the total length of a straight line (geodesic) is $2\pi R$, and in the latter πR; the planes are two-sided, and one-sided, respectively, and so on.

of the x will follow by themselves. There is no need to declare beforehand that they are cartesian coördinates of a place and its date. Further, the circumstance that these coefficients are of different signs, three being negative, and one positive, creates for the general geometer no difficulty.

This circumstance brings only with it the important feature that there are in the world real *minimal lines*,* as the geometer would put it, that is to say, lines of zero-length,

$$ds=0,$$

or

$$\frac{dx_1^2+dx_2^2+dx_3^2}{dx_4^2} = \frac{1}{c^2}\left(\frac{d\sigma}{dt}\right)^2=1.$$

These special world lines represent the propagation of light or, apart from physical difficulties, the uniform motion of a particle with light velocity c. As a matter of fact the very first step of the theory consisted in writing $ds=0$ as the expression of light propagation in vacuo.

In the next place consider the equally fundamental concept of *the geodesics* of the world. These are defined by

$$\delta\int ds=0,$$

the limits of the integral being kept fixed. To derive from this variational equation the differential equations of a geodesic, proceed in the well-known way. If u be any independent parameter, and if dots are used for the derivatives with respect to it, we have

$$\delta\int \dot{s}du=\int\delta\dot{s}\,.\,du=0,$$

where, by (1), $\dot{s}^2=-(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2)+\dot{x}_4^2$, and therefore,

$$\delta\dot{s} = \frac{1}{\dot{s}}\,(\dot{x}_4\delta\dot{x}_4-\dot{x}_1\delta\dot{x}_1-\dot{x}_2\delta\dot{x}_2-\dot{x}_3\delta\dot{x}_3).$$

*Whereas on any (real) surface all the 'minimal lines' (known also as null-lines), which play in the surface theory an important analytical rôle, are always imaginary. The reader will do well to consult on this and allied topics a special treatise on differential geometry.

By partial integration, and remembering that all δx_ι vanish at the limits of the integral,

$$\int \frac{1}{\dot{s}}\,\dot{x}_\iota \delta\dot{x}_\iota\,.\,du = -\int \frac{d}{du}\left(\frac{1}{\dot{s}}\,\dot{x}_\iota\right)\delta x_\iota\,.\,du.$$

Thus, the δx_ι being mutually independent, the required differential equations are

$$\frac{d}{du}\left(\frac{\dot{x}_\iota}{\dot{s}}\right)=0.$$

If the geodesic does not happen to be a null-line (light propagation) we can as well take $u=s$, when $\dot{s}=1$, and the equations become

$$\frac{d^2x_\iota}{ds^2} = 0,$$

whence

$$\frac{dx_\iota}{ds} = \frac{dx_\iota}{dx_4}\frac{dx_4}{ds} = a_\iota = \text{const.}$$

The fourth of these equations is $dx_4/ds=\text{const.}$, and therefore, the first three,

$$\frac{dx_1}{dt} = a_1,\ \frac{dx_2}{dt} = a_2,\ \frac{dx_3}{dt} = a_3,$$

and these represent uniform rectilinear motion, which is the motion of a free particle.

Let us, therefore, keep well in mind these two properties of the line-element ds of special relativity:

I. *The minimal lines of the world,*

$$ds=0, \tag{I}$$

represent light propagation in vacuo.

II. *The world geodesics, defined by*

$$\delta\int ds=0, \tag{II}$$

with fixed integral limts, represent the motion of a free particle.

A special emphasis is here put on these two properties because they will be carried over to the general relativity and gravitation theory, and because these and principally only these two properties constitute the connection of the otherwise purely analytical differential form $ds^2 = g_{\iota\kappa} dx_\iota dx_\kappa$ with physics. In other words, (I) as the equation of light propagation, and (II) as that of the motion of a free particle impart physical meaning to the mathematical form which is the 'line-element' ds. Without this all the properties of the quadratic form, though interesting, perhaps, in themselves, would have nothing to do with the world of physical phenomena.

It is scarcely necessary to say that the law (II) of the motion of free particles is, as well as (I) for light, invariant (thus far) with respect to the Lorentz transformation. For it is, by its very structure, independent of the choice of a reference system S. Since ds is invariant, so is $\int ds$, extended between any two world-points. Thus also the developed form of (II), the system of differential equations $d^2x_\iota/ds^2 = 0$, is transformed in S' into $d^2x_\iota'/ds^2 = 0$. And in fact, uniform motion of a particle relatively to S, means also (originally by an assumption) its uniform motion in any other inertial system S'. In short, the Lorentz transformation leaves the uniformity of motion of a particle intact.

5. We are now ready to pass to Einstein's theory of general relativity and gravitation. Not that our task is an easy one, but we are somewhat better prepared to embark upon it.

Why equal form of physical laws, why equal rights for the inertial systems only? Why not equal rights for all (systems)? Such would be the urgent, and yet vague, questions naturally suggesting themselves after what was said in the preceding sections. Yet it is not with these questions, nor with an attempt to answer them, that we will begin our journey across this new and revolutionary country. For, even if answered, these questions would remain physically barren were it not for the existence of gravitation and

especially of a certain peculiarly simple property of this universal agent.

This, therefore, will first occupy our attention for a while. The cardinal feature of gravitation just hinted at is the proportionality of weight to mass, in other words, the proportionality of *heavy* (gravitating) and *inert* mass. First tested by Newton in his famous pendulum experiments with bobs of different material, and carried to further precision by Bessel, this proportionality has been more recently shown by Roland Eötvös to hold to one part in ten millions. It is reasonable, therefore, to assume, with Einstein, that it holds rigorously,* at least until proofs to the contrary are forthcoming. In our present connection it is better to express this property more directly by saying, even with Galileo, that all bodies, light or heavy, fall equally in vacuo. All particles, that is, acquire at a given place of a gravitational field *equal accelerations* independently of their own mass or chemical nature, etc., and no matter how much of their inertia is due to the energy stored in them and how much of other origin. This remarkable property distinguishes the gravitational field from other fields. Take, for instance, an electric field given by the vector $\mathbf{E}$. The force on a particle of rest-mass m, carrying the electric charge e, and starting from rest, is $e\mathbf{E}$, and the acceleration $e\mathbf{E}/m$. Now, in general, there is no relation between m and e, and even if the mass is purely electromagnetic, when m is proportional to e^2/a, the acceleration will vary from particle to particle inversely as its charge and directly as its average diameter, $2a$. We have disregarded, of course, the dielectric properties of the particle which would make its behaviour in a given electric field still more complicated. The same remarks would hold, *mutatis mutandis*, for the behaviour of different bodies placed in a magnetic field. In short, gravitation is, in this respect, unique in its simplicity.

*In a theory of matter and gravitation proposed by G. Mie, *Annalen der Physik*, vols. 37, 39, 40 (1912 and 1913), the proportionality between weight and mass does not hold rigorously, though to an order of precision much exceeding that stated by Eötvös.

This very circumstance enabled Einstein to undertake his mental experiment with the falling or ascending elevator, now so familiar to the general public. In fact, consider a homogeneous or a quasi-homogeneous gravitational field such as the terrestrial one in a properly restricted region. Let a lift or elevator, small compared with the earth, yet ample enough for a physical laboratory and for those in charge of it, descend vertically with the local terrestrial acceleration g. Then all bodies placed anywhere within the elevator and left to themselves will float, in mid-air or better in vacuo, and particles projected in any direction will move uniformly in straight paths relatively to the elevator. Moreover, all objects, including the physicists, standing or lying about will cease to press against the floor or the tables, as the case may be. In short, all traces of gravitation will be gone,* and the inmates of the lift, assumed to have no intercourse whatever with the outer world, will declare their reference system to be a genuine inertial system,—so far, at least, as mechanical phenomena are concerned. For an unbiassed judge could not tell beforehand whether it will be also optically inertial, that is to say, whether the law of constant light velocity will hold good for the lift. Einstein thinks it will, or rather assumes it, more or less implicitly. If this be granted, we can say that the elevator will be an inertial reference system in every respect.

The possibility of thus undoing a gravitational field is manifestly based on the said equal behaviour of all bodies placed in it. For otherwise the artificial motion of the elevator could not be adapted to all bodies at the same time, each of them requiring a different acceleration.

Next, pass to any, non-homogeneous gravitation field, which in the most general case may also vary with time. This certainly cannot be undone, as a whole, by a single elevator as reference system. But you can imagine an ever increasing number of sufficiently small elevators, each appropriately accelerated, fitted into small regions of the field, and each,

* *Vice versa*, in absence of a gravitational field, a lift in accelerated ascending motion would give us a faithful imitation of such a field.

perhaps, to do its duty for a very short interval of time, and to be replaced by another in the next moment. These minute elevators will do their office at least in the mechanical sense of the word. Einstein assumes that they will act as inertial systems also in the optical sense of the word, as explained above. This process of subdividing a gravitational field, in space and time, and fitting in of appropriate small elevators can be carried on to any required degree of approximation.

In fine, passing to the limit, let us make, with Einstein,* the explicit assumption:

With an appropriate choice of a local reference system (u_1, u_2, u_3, u_4) *special relativity holds for every infinitesimal four-dimensional domain or volume-element of the world.*

That is to say, at every world-point† a system of space-time coördinates u_1, u_2, u_3, u_4 can be chosen in which the line-element assumes the galilean form

$$ds^2 = du_4^2 - du_1^2 - du_2^2 - du_3^2. \tag{3}$$

These four coördinates are called *local coördinates.* With respect to this local system there is then no gravitational field at the given world-point, and in accordance with special relativity ds^2 has there a value independent of the 'orientation' of the local axes; that is to say, the quadratic form (3) is invariant with respect to the Lorentz transformation (2).

It is this assumption which can now be properly referred to as the *infinitesimal equivalence hypothesis*, for it grew out of Einstein's original equivalence hypothesis applied to finite regions when, in his first attempt at a theory of gravitation (1911), he was confining himself to a homogeneous field.

Whatever the origin of this hypothesis or assumption, it is certainly not difficult to adhere to it. For it scarcely amounts to anything more than to assuming, in the case of a curved surface, say, the existence of a tangential plane at any of its

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*A. Einstein, *Die Grundlagen der allgemeinen Relativitätstheorie. Annalen der Physik*, vol. 49, 1916, p. 777.

†With the possible exception of some discrete points, such perhaps as those at which the density of matter acquires enormous values.

points, or to declare the surface to be (in Clifford's terminology) elementally flat. And it will, perhaps, be well to restate shortly Einstein's hypothesis by saying that it assumes *the four-dimensional world* to be, in presence as well as in absence of gravitation, *elementally flat*. It will not be forgotten, however, that this geometric term is nothing more than a synonym of *elementally galilean*, *i.e.*, satisfying special relativity infinitesimally.*

To avoid the danger of any misconception let us dwell upon this subject yet for a while. The coördinates u_ι with their corresponding galilean line-element (3) were set up only for a local purpose, their real office being confined to a fixed world-point P, say x_1, x_2, x_3, x_4 (in any coördinate system). If we so desire, we may think of a whole galilean world U determined throughout, to any extent, by the simple form (3). But as a tangential plane has something in common with a surface only *at* the point of contact and then diverges from it, ceasing to represent any intrinsic properties of the surface itself, so has the auxiliary and fictitious world U anything to do with the actual world W (complicated by gravitation) at the world point x_ι only. The fictitious world U is *tangential* to the actual world W at that point, and parts company with it beyond the point of contact. At other world-points the role of U is taken over by other and other fictitious galilean worlds. One more cautious remark. The contact of U and W is one of *the first order*, *i.e.*, such as the contact between a surface and its tangential plane or between a curve and its tangential line, but not as the more intimate contact bewteen a curve and its circle of curvature (which is of the second order). This circumstance may acquire some importance later on.

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*As to the concept of elementary flatness of a surface or a more-dimensional space, it is beautifully explained in W. K. Clifford's 'Philosophy of Pure Sciences', published in his famous Lectures and Essays (Macmillan, London). Notice that in Clifford's sense every regular surface, no matter how curved, is elementally flat, with the exception of some singular points, such as the vertex of a cone.

6. Having thus made clear the local character of the u_ι coördinates, let us now introduce any coördinate system x_ι whatever, to be used as a reference system of coördinates for the whole world, *i.e.*, throughout the gravitational field and through all times. Then, if x_ι be the reference coördinates of P, and $x_\iota + dx_\iota$ those of a neighbour-point $u_\iota + du_\iota$, the differentials du_ι will in general be linear homogeneous functions of all the dx_ι, say

$$du_\iota = \sum_{\kappa=1}^{4} a_{\iota\kappa}\, dx_\kappa,$$

or with the conventional abbreviation,

$$du_\iota = a_{\iota\kappa}\, dx_\kappa, \tag{4}$$

where the coefficients $a_{\iota\kappa}$ will in general be functions of all the x_ι. It is of importance to note that that the relations (4) will, generally speaking, be not-integrable, or borrowing a name from dynamics, *non-holonomous*, that is to say, the $a_{\iota\kappa}$ will not necessarily be $\partial u_\iota / \partial x_\kappa$, the differential expressions on the right of (4) will not be total differentials of functions of the x_ι, and there will be no finite relations between the local and the general or reference-coördinates.

Substituting (4) into (3), collecting the terms and calling $g_{\iota\kappa}$ the coefficient of the product of differentials $dx_\iota dx_\kappa$ we shall have, for the line-element in the general reference system,

$$ds^2 = g_{\iota\kappa}\, dx_\iota\, dx_\kappa, \tag{5}$$

where $g_{\iota\kappa} = g_{\kappa\iota}$ will, in the most general case, be functions of all the x_ι. But since ds^2, as defined originally by (3), was independent of the orientation of the local system of axes, so also will the ten different coefficients $g_{\iota\kappa}$, though functions of the coördinates x_ι, be manifestly independent of the orientation of the local system.

The line-element will thus be represented, in any reference coördinates x_ι whatever by *the most general* quadratic differential form of these four variables, such in fact being the form (5). As before the summation sign is omitted; the summation is to be extended over ι, κ from 1 to 4, each of these

suffixes, ι and κ, appearing twice. Thus, $ds^2 = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + \ldots + g_{44}dx_4^2$. The reader will soon learn to handle this abbreviated and very convenient symbolism.

Suppose now we introduce instead of x_ι any other set of space-time coördinates x_ι', any functions whatever of the x_ι, such, that is, that between the two sets exist any given holonomous relations

$$x_\iota = \phi_\iota (x_1', x_2', x_3', x_4'), \tag{6}$$

the ϕ_ι being any functions whatever, but continuous together with their first derivatives and such that their Jacobian, the well-known determinant

$$J = \left| \frac{\partial x_\iota}{\partial x_\kappa'} \right| = \begin{vmatrix} \frac{\partial x_1}{\partial x_1'} & \frac{\partial x_1}{\partial x_2'} & \cdots & \frac{\partial x_1}{\partial x_4'} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial x_4}{\partial x_1'} & \frac{\partial x_4}{\partial x_2'} & \cdots & \frac{\partial x_4}{\partial x_4'} \end{vmatrix} \tag{7}$$

does not vanish. Under these circumstances we have

$$dx_\iota = \frac{\partial x_\iota}{\partial x_\kappa'} dx_\kappa', \tag{8}$$

and *vice versa*,

$$dx_\iota' = \frac{\partial x_\iota'}{\partial x_\kappa} dx_\kappa, \tag{8a}$$

and, as it may be well to notice in passing, $JJ' = 1$, where J' is the inverse Jacobian $\left| \frac{\partial x_\iota'}{\partial x_\kappa} \right|$. Now, substituting (8) into (5), gathering again the terms, and denoting the coefficient of $dx_\iota'dx_\kappa'$ by $g_{\iota\kappa}'$, *i.e.*, putting

$$g_{\iota\kappa}' = \frac{\partial x_\alpha}{\partial x_\iota'} \frac{\partial x_\beta}{\partial x_\kappa'} g_{\alpha\beta}, \tag{9}$$

we shall have

$$ds^2 = g_{\iota\kappa}' \, dx_\iota' \, dx_\kappa' \tag{5'}$$

which is (5) reproduced in dashed letters. Not that the $g_{\iota\kappa}'$ will be functions of the x_ι' of the same form as were the $g_{\iota\kappa}$ of

x_ι, but only that the quadratic differential form remains quadratic. There is certainly nothing surprising in this kind of permanence.* Yet this and this only is justifiably meant when we say that the line-element ds^2 is invariant with respect to any transformations whatever. If the relativist sees anything more in "the invariance of ds", namely that ds is something belonging to a pair of world-points (x_ι and $x_\iota + dx_\iota$) inherent in that pair independently of the choice of a reference system, it is what he puts into it at a later stage by ascribing to it certain physical properties, or by interpreting it physically in certain ways. The meaning of these remarks will gradually become more intelligible.

Before passing on to the two cardinal virtues conferred upon the line-element, one more mathematical remark about it may not be out of place just here. Suppose the line-element (5) is actually given with some determined and more or less complicated functions as the $g_{\iota\kappa}$. By trying, in succession, other and other new variables x_ι' we would arrive at a great variety of new forms of functions $g_{\iota\kappa}'$. The natural question arises: Are there not among all these sets of coordinates just such as would convert (5), throughout the world or a finite world-domain, into a galilean line-element, *i.e.*, one with constant coefficients? The answer is, in general, in the negative. A given form $ds^2 = g_{\iota\kappa}\, dx_\iota\, dx_\kappa$ is *equivalent*, that is to say, can be reduced by *holonomous* transformations, to a form with constant coefficients and thus also to the galilean line-element† when and only when certain differential expressions formed of the $g_{\iota\kappa}$, their first and second derivatives, all vanish.* These expressions, of which more will be

*Notice that the case is different in special relativity, where we require the form to reappear with all its original coefficients, three -1, and one $+1$.

†The circumstance that three of the coefficients of this form are negative and one positive imposes on the original $g_{\iota\kappa} dx_\iota dx_\kappa$ to be thus transformable certain further conditions in connection with the so-called 'law of (algebraic) inertia', due to Sylvester.

*The restriction to '*holonomous* transformations' is of prime importance. For by means of non-holonomous or non-integrable relations, such as (3), every $g_{\iota\kappa}\, dx_\iota\, dx_\kappa$ can be transformed into a quadratic differential form with constant coefficients.

said in the sequel, are known in general differential geometry as Riemann's four-index symbols. Of these symbols there are in the case of any number n of dimensions, $\frac{1}{12}n^2(n^2-1)$ linearly independent ones. Thus an ordinary, two-dimensional, surface has but one Riemann symbol and this is its *Gaussian curvature*, multiplied by $g_{11}g_{22}-g_{12}^2$, the determinant of the $g_{\iota\kappa}$. Any three-dimensional manifold has *six*, and our, or rather Einstein-Minkowski's world has as many as *twenty* linearly independent Riemann symbols. Thus any finite domain of the world is equivalent to a galilean domain when and only when all these twenty symbols vanish in that domain, *i.e.*, when the *ten* different $g_{\iota\kappa}$ satisfy within it a system of *twenty* partial differential equations of the *second order*. (It will be useful to keep in mind the last italics.) By what has just been said it is manifest that if all the Riemann symbols vanish in one system of coördinates x_ι, they will vanish also in any other x_ι' obtained from the former by any holonomous transformations whatever.

But enough has for the present been said on the symbols of that great geometer. Later on they will be seen to play an all-important role in Einstein's gravitation theory.

It is now time to return to the physical aspect of our subject.

7. Having assumed, after Einstein, that special relativity holds for every infinitesimal domain, or that the world is elementally galilean, we wrote down the simple form (3) in local coördinates u_ι. Then, passing to any coördinates x_ι by means of the non-holonomous relations (4) we obtained for the line-element of the world the general quadratic differential form (5), with variable coefficients $g_{\iota\kappa}$, functions of the x_ι.

But what is the physical meaning of this general ds with all its ten different $g_{\iota\kappa}$? What are they to represent physically? The answer is that we are still to a certain extent the masters of the situation, and can make them have that physical meaning which we will put into them. For thus far we know only the physical meaning of the galilean element belonging to a world U, and that (in virtue of an assumption) the world

W as a seat of or deformed by gravitation is galilean in its elements, or that at each of its points a U-world tangential to it can be constructed.

At this stage then we are entitled only to say that (since W without gravitation is U, and since to U belong the constant coefficients $\bar{g}_{\iota\kappa}$) the essential differences* in the coefficients of the two worlds, $g_{\iota\kappa}$ and $\bar{g}_{\iota\kappa}$, are due to, or better, are *somehow* connected with gravitation. But exactly how, we cannot, thus far, say. For our position is somewhat like this: Suppose we know that a surface σ, which is not a plane as a whole, is elementally flat and thus has a tangential plane π at each of its points. Suppose further we know the physical properties of certain lines (straights, or circles, etc.) drawn on any π. Does this alone enable us to say what the physical properties of similarly defined lines will be when drawn on σ? Clearly not. For the π-lines have but a single point of contact with σ, and that only of the first order, and deviate from the surface or become *extra-σ* beings all around the point of contact.

Now, in the case of space-time, we fixed the physical meaning of the line-element of the U-world by declaring its minimal lines, $ds=0$, to be the law of light propagation, and its geodesics, $\delta\int ds=0$, to represent the motion of free particles. Does this, and the existence of a tangential U at every point of the actual world W, entitle us to assert that the minimal lines and the geodesics of W will again represent the optical and the mechanical laws in this world? This is by no means a superfluous question. For the auxiliary tangential world U leaves the actual world beyond the point of contact and becomes at once fictitious or extra-mundane, so to speak.

Now, the minimal lines of U,† defined by a differential equation of *the first order*, are also, at P, minimal lines of W, so that at least the starting elements of these lines are identical. At the next element the rôle of U is taken over by another

i.e., those, at least, which cannot be abolished by holonomous co-ordinate transformations.

†Which fill out only a conic hypersurface (of three dimensions) with the contact point P as apex.

galilean world; yet the reasoning can be repeated, so that we can say that every element of a minimal line of W represents light propagation, and thence deduce that such a W-line possesses also as a whole the same physical property. But the position is altogether different with the geodesics. For these world-lines are defined by differential equations of *the second order*.* so that the mere contact of U and W (being of the first order) does not at all entitle us to transfer any properties of the geodesics of U upon those of W, not even at their very starting point P.

If, however, the said physical property of the W-geodesics does not follow logically from the previous assumptions, yet we are free to introduce it as a further explicit assumption. In fact, while thus generalizing the physical significance of the geodesics Einstein is well aware that this is a new assumption,† although one that easily suggests itself. Nor is there any inconsistency in thus transfering a property from the galilean to the more general world-geodesics. For, as we shall see later on, the developed form of the equations of the geodesics contains only the $g_{\iota\kappa}$ and their first derivatives with respect to the x_ι, whereas the conditions characterizing a world as galilean (the vanishing of the Riemann symbols) are equations between the $g_{\iota\kappa}$, their first *and second* derivatives, and there are no relations at all between the $g_{\iota\kappa}$ and their first derivatives alone.

But even with this new assumption, the total number of assumptions of Einstein's theory is remarkably small. And as to the advisability of making the one just discussed, we may say that Einstein's theory owes to it the greater part of its power.

The property of the geodesics being thus assumed, and that belonging to the minimal lines being deducible from what preceded, we are now in the position to sum up definitely and very concisely, if not the whole, yet the most fundamental part of Einstein's theory. For this purpose we have only to

*A geodesic issues from P in *every* direction whatever in the four-manifolds U and W.

†A. Einstein, *loc. cit.*, p. 802.

repeat the previous statements I. and II. without their restrictions, replacing the galilean ds by the general one and adding a few explanatory words. Thus:

The world-line element, in any system of coördinates, and whether gravitation be absent or present, is given by

$$ds^2 = g_{\iota\kappa}\, dx_\iota\, dx_\kappa, \tag{10}$$

where $g_{\iota\kappa} = g_{\kappa\iota}$ *are some functions of, in general, all the four coördinates, but of these alone. If these ten functions be given, all metrical properties* of the world are determined, and among these its minimal lines,*

$$ds = 0, \tag{I}$$

and its geodesics,

$$\delta \int ds = 0. \tag{II}$$

The physical significance of these world-lines is that the former represent propagation of light in vacuo, and the latter the motion of a free particle.

By a 'free' particle is meant one which, having received any initial impulse is left to its own fate, whether in absence or in proximity of other lumps of matter (absence or presence of 'gravitation'), but not colliding with them, and in absence of, or better not immersed in, an electromagnetic field. One strives in vain to enumerate all the attributes of a concept which can become clear only *a posteriori*, through the concrete applications of the theory. Suffice it to say that 'free particle' may as well stand for a projectile, in vacuo, or a planet circling around the sun. Their laws of motion are given by the corresponding world-geodesics. The developed form of the equations of the geodesics, as well as of light propagation, will be given later on.

Since the $g_{\iota\kappa}$ are to determine, through (II), the fall of projectiles and the motion of celestial bodies, it is scarcely necessary to repeat that they are intimately connected with gravitation. These ten coefficients will replace the unique scalar potential of newtonian mechanics. They will influence

*Apart from some properties of the world as a whole,—of which more later on.

also, through (I), the course of light in interplanetary and interstellar spaces, and finally, by their very appearance in the line-element, they will mould the geo- and chrono-metrical properties of our world. These latter properties thus appear intimately entangled with gravitation and optics.

It remains to explain how these all-powerful coefficients are, in their turn, determined in terms of other things such as the density of 'matter'. This is the office of Einstein's 'field-equations' which will occupy our attention in the sequel.

CHAPTER II.

The General Relativity Principle. Minimal Lines and Geodesics. Examples. Newton's Equations of Motion as an Approximation.

8. Most readers will perhaps be surprised to find in the first chapter almost no mention of the general principle of relativity which claims equal rights for all systems of co-ordinates, and which in all publications on our subject is given the most prominent place. Instead of this we insisted on the general form of the line-element (10), on the null-lines and the geodesics of the world metrically determined by that line-element, and still more upon the physical meaning of these two kinds of world-lines as representing light propagation and the motion of free particles.

The reason for adopting this plan is that, as far as I can see, these things are most important from the *physical* point of view, nay, they are perhaps* the only relevant constituents of the new theory looked upon as a physical theory. This is particularly true of the optical and mechanical meaning attributed to the said two kinds of lines, thus giving what the logicians call a *concrete representation* of what otherwise would be only a purely mathematical or logical science, an abstract geometry of a manifold of four dimensions determined by that quadratic differential form. It is exactly this physical interpretation which invests the theory with the power of making statements of a phenomenal content, of predicting the course of observable events. On the other hand, the much extolled Principle of General Relativity which, in Einstein's wording,† requires
The general laws of Nature to be expressed by equations valid

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*Apart from 'the field equations', yet to come.

†*Loc. cit.*, p. 776.

in all coördinate systems, i.e., covariant with respect to any substitutions whatever (generally covariant),
is by itself powerless either to predict or to exclude anything which has a phenomenal content. For whatever we already know or will learn to know about the ways of Nature, provided always it has some phenomenal contents (and is not a merely formal proposition), should always be expressible in a manner independent of the auxiliaries used for its description. In other words, the mere requirement of general covariance does not exclude any phenomena or any laws of Nature, but only certain ways of expressing them. It does not at all prescribe the course of Nature but the form of the laws constructed by the naturalist (mathematical physicist or astronomer) who is about to describe it. The fact that some phenomenal qualities are technically (with our inherited mathematical apparatus) much more difficult to put into a generally covariant form than some others does not in the least change the position.

To make my meaning plain, let us take the case of planetary motion. For the sake of simplicity let there be but a single planet revolving around the sun. It is well-known that according to Newton the orbit of the planet should be a conic section, say an ellipse with fixed perihelion.* It is, in our days, almost equally well known that according to Einstein's theory the perihelion should move, progressively, showing a shift at the completion of each of its periods. And so it does, at least to judge from Mercury's behaviour. At the same time Einstein's equations are generally covariant, while Newton's 'law' or Laplace-Poisson's equations are not.† What of this? Does it mean that fixed perihelia are excluded or prohibited by the principle of general covariance? Certainly not. Provided that 'fixed perihelion' and 'moving perihelion' have, each, a phenomenal content, and this they do, both kinds of planetary behaviour should be expressible in a generally covariant form. Newton's inverse square law and his equations of motion certainly do not express it so,

*Fixed, that is, relatively to the stars.

†Not even with respect to the special or the Lorentz transformation.

and it may be difficult to find a covariant expression for a strictly keplerian behaviour. But if it were urgently needed, some powerful mathematician would, no doubt, succeed in constructing it. If, as actually is the case, Einstein's theory excludes a fixed perihelion, and other newtonian features, it does this not in virtue of the said principle alone (nor even in part), but pre-eminently owing to the physical meaning ascribed to the world-geodesics, and to the choice of his field equations which again are physically relevant since they determine the $g_{\iota\kappa}$ influencing essentially the form of those world-lines. That the principle of general relativity turned out to be helpful in guessing new laws (by limiting the choice of formulae) is an altogether different matter. It may prove an even more successful guide in the future.† But here its rôle ends,—always taking the Principle only as a *mathematical* requirement of general covariance of equations. And so it is at any rate enunciated (and interpreted, cf, p. 776, *loc. cit.*) by Einstein himself, although some of his exponents put into it a physical meaning. In fact, as we shall see later on, the sameness of form of the equations (of motion, say) in two reference systems, as in a smoothly rolling and a vehemently jerked car, does not at all mean sameness of phenomenal behaviour for the passengers of these two vehicles.

So much in explanation of the absence of the general principle of relativity in all our preceding deductions.

It will be noticed, however, that although no explicit mention of this principle has been made in Chapter I, yet the fundamental laws (I) and (II) there given do satisfy this principle. In fact, both the null-lines and the geodesics of the world were defined without the aid of any reference system. And as to the line-element itself, its invariance was seen to be automatic.

Thus, in what precedes we have, without insisting upon it, been faithful to the formal principle of general relativity. Nor is it our intention to depart from it in what will follow.

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†Or it may become sterile to-morrow, as is the fate of almost all our Principles.

As was already mentioned at the close of the first chapter, to make the exposition of the fundamental part of Einstein's theory complete, it remains to add to (10), (I), (II), together with their optical and mechanical meaning, a set of equations determining the ten coefficients $g_{\iota\kappa}$ of the quadratic form. But before passing to these differential equations, Einstein's field-equations, it will be well to discuss somewhat more and to develop those already given. Some explanations and examples concerning the transformation of coördinates will also be helpful at this stage.

9. First, concerning the law of propagation of light (in vacuo), to obtain its developed form it is enough to substitute the line-element (10) into the equation (I) of the minimal lines. Thus the fundamental optical law will be

$$g_{\iota\kappa}\,dx_\iota\,dx_\kappa = 0. \tag{11}$$

It gives *the velocity of light* for every direction of the ray, *i.e.*, of the infinitesimal space-vector dx_1, dx_2, dx_3, if dx_4/c be the time element of the reference system. In general the light velocity will differ from c and have different values at different world points and for different directions of the ray.

This "light velocity" which has nothing intrinsic about it is to be distinguished from the *local* velocity of light (that corresponding to a local, galilean system of coördinates) which is the same for all directions. To avoid confusion the former may be called the system-velocity of light or, according to some authors, the 'coördinate velocity' of light. It is a kind of velocity estimated from a distant standpoint. If we write it, in a given reference system,

$$\frac{d\sigma}{dt} = c\frac{d\sigma}{dx_4},$$

the very concept of such a light velocity, whose value is to be derived from (11), presupposes that 'the length' $d\sigma$ of the infinitesimal space-vector dx_1, dx_2, dx_3 has been defined in some way for that system in terms of these differentials and the coefficients $g_{\iota\kappa}$. We shall have the best opportunity of

explaining how this is done technically in deducing physical results, when we come to speak of the bending of rays of light around a massive body such as the sun. Then also the question will be mentioned under what circumstances the law of Fermat, giving the shape of the rays, is applicable.

In the meantime it is advisable to look upon (11) as the equation of the infinitesimal *wave surface* at the instant $t+dt$ corresponding to a light disturbance started at x_1, x_2, x_3 at the instant t, the differential dx_4 being treated as a constant parameter. From the local standpoint this surface is, of course, a sphere, but from the distant (or system-) standpoint it may have a variety of more complicated shapes. It would, perhaps, be rash to say that it will be a quadric. But, being locally closed, it may also be expected to be a closed surface from the system-point of view.

10. Next for *the geodesics* of the world. The developed form of their differential equations is easily derived from their original definition (II),

$$\delta \int ds = 0.$$

As in the case of a galilean world, let u be any parameter, and let dots stand for derivatives with respect to it. Then

$$\int \delta \dot{s} \, . \, du = 0,$$

where, in the most general case,

$$\dot{s}^2 = g_{\iota\kappa} \dot{x}_\iota \dot{x}_\kappa. \tag{12}$$

The variation of $\dot{s}$ can be written

$$\delta \dot{s} = \frac{\partial \dot{s}}{\partial x_\iota} \delta x_\iota + \frac{\partial \dot{s}}{\partial \dot{x}_\iota} \delta \dot{x}_\iota,$$

to be summed over $\iota = 1$ to 4. Thus, by partial integration of the second terms, the limits of the integral being fixed,

$$\frac{d}{du}\left(\frac{\partial \dot{s}}{\partial \dot{x}_\iota}\right) - \frac{\partial \dot{s}}{\partial x_\iota} = 0,$$

and by (12), with s itself taken for u,

$$\frac{d}{ds}\left(g_{\iota\kappa}\frac{dx_\kappa}{ds}\right) - \tfrac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x_\iota}\cdot\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0,$$

or

$$g_{\iota\kappa}\frac{d^2x_\kappa}{ds^2} + \frac{\partial g_{\iota\kappa}}{\partial x_\lambda}\frac{dx_\lambda}{ds}\frac{dx_\kappa}{ds} - \tfrac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x_\iota}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0.$$

Introducing the expressions, known as Christoffel's symbols,

$$\begin{bmatrix}\alpha\beta\\ \gamma\end{bmatrix} = \tfrac{1}{2}\left(\frac{\partial g_{\gamma\alpha}}{\partial x_\beta} + \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\gamma}\right) = \begin{bmatrix}\beta\alpha\\ \gamma\end{bmatrix}, \tag{13}$$

we can condense the last set of equations into

$$g_{\iota\kappa}\frac{d^2x_\kappa}{ds^2} + \begin{bmatrix}\alpha\beta\\ \iota\end{bmatrix}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0.$$

These are four linear equations for the four d^2x_κ/ds^2. Let us solve them for these derivatives. Denoting the second term by a_ι, and writing g for the determinant of the $g_{\iota\kappa}$, we shall have

$$\frac{d^2x_1}{ds^2} + \frac{1}{g}\begin{vmatrix} a_1 & g_{12} & g_{13} & g_{14} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_4 & g_{42} & g_{43} & g_{44}\end{vmatrix} = 0, \text{ etc.}$$

or, if $g^{\iota\kappa} = g^{\kappa\iota}$ be the minor of g, corresponding to $g_{\iota\kappa}$, divided by g itself,

$$\frac{d^2x_1}{ds^2} + a_1g^{11} + a_2g^{12} + a_3g^{13} + a_4g^{14} = 0, \text{ etc.},$$

i.e.,

$$\frac{d^2x_\iota}{ds^2} + g^{\iota\kappa}\begin{bmatrix}\alpha\beta\\ \kappa\end{bmatrix}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0.$$

Here we will write, after Christoffel,

$$\begin{Bmatrix}\alpha\beta\\ \iota\end{Bmatrix} = g^{\iota\kappa}\begin{bmatrix}\alpha\beta\\ \kappa\end{bmatrix} = \begin{Bmatrix}\beta\alpha\\ \iota\end{Bmatrix}. \tag{14}$$

Thus, ultimately, the differential equations of the geodesics or *the equations of motion of a free particle* will be, in any system of coördinates,

$$\frac{d^2x_\iota}{ds^2}+\left\{\begin{matrix}\alpha\beta\\ \iota\end{matrix}\right\}\frac{dx_\alpha}{ds}\,\frac{dx_\beta}{ds}=0.* \tag{15}$$

These are four equations. But since we have, identically,

$$g_{\iota\kappa}\frac{dx_\iota}{ds}\,\frac{dx_\kappa}{ds}=1,$$

one of these equations of motion is a consequence of the remaining three, a feature already familiar to the reader from special relativistic mechanics. Since these differential equations are only the developed form of $\delta\int ds=0$, they will manifestly be generally covariant, that is to say, in any new coördinates x_ι' the equations (15) will be

$$\frac{d^2x_\iota'}{ds^2}+\left\{\begin{matrix}\alpha\beta\\ \iota\end{matrix}\right\}'\frac{dx_\alpha'}{ds}\,\frac{dx_\beta'}{ds}=0.$$

If the coefficients $g_{\iota\kappa}$ are all constant, all the Christoffel symbols $\left\{\begin{matrix}\alpha\beta\\ \iota\end{matrix}\right\}$ vanish and the equations (15) reduce to $d^2x_\iota/ds^2=0$, which represent uniform rectilinear motion. And since the general equations (15) represent the motion of a free particle in any gravitational field and in any system, the symbols $\left\{\begin{matrix}\alpha\beta\\ \iota\end{matrix}\right\}$, built up of the $g_{\iota\kappa}$ and their first derivatives, can be said to express the deviation of the motion from uniformity due to gravitation, and partly due to the peculiarities of the system of reference. In view of this property, and disregarding any distinction between gravitation proper and the effects of the choice of the coördinate system,† Einstein

*This form of the equations of a geodesic of a manifold, of any number of dimensions, has been used by geometers for a long time. See, for instance, L. Bianchi's *Geometria differenziale*, vol. I, Pisa 1902, p. 334.

†Or between permanent acceleration fields and such that can be transformed away.

proposes to call these Christoffel symbols 'the components of the gravitational field'.

Notice, however, that if all $\left\{ \begin{matrix} \alpha\beta \\ \iota \end{matrix} \right\}$ vanish in one system of reference they *do not* necessarily vanish in other systems* (even if obtained from the former by holonomous transformations). In view of this circumstance the name proposed by Einstein seems utterly inappropriate and misleading, even if one agreed not to distinguish between permanent fields and such that can holonomously be transformed away, as for instance the 'centrifugal force'.

10a. In fact, consider for example the galilean line-element in three dimensions, *i.e.*, for $\phi = \text{const.} = \pi/2$,

$$ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2,$$

taking ct, r, θ as x_4, x_1, x_2 respectively. Calculate the corresponding Christoffel symbols. Since $g_{11} = -1$, $g_{22} = -r^2$, $g_{44} = 1$, and all other $g_{\iota\kappa}$ vanish, we have, for instance, the non-vanishing symbol

$$\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} = -r.$$

But who would call it a 'component of the gravitational field'? This case is a particularly drastic one, for the world-geodesics corresponding to our line-element do represent uniform rectilinear motion. The appearance of non-vanishing Christoffel symbols is simply due to the use of polar instead of cartesian co-ordinates.

In short, gravitation certainly contributes to the Christoffel symbols, but so does also a mere transformation of space-coördinates, although it has nothing whatever in common with 'gravitation' of the permanent or the non-permanent kind. This criticism does not in the least diminish the value of the general equations of motion (15). It is given here only to prevent misconceptions which have seemed particularly likely in the case of beginners.

*In the terminology of the tensor calculus, to be explained later on, the Christoffel symbols are *not* the components of a *tensor*.

10b. Let us take yet another simple example, this time not for the sake of criticism but because of its instructiveness.

Consider the line-element arising from the galilean one, just quoted,

$$(S') \qquad ds^2 = dx'^2_4 - dr'^2 - r'^2 d\theta'^2,$$

by the transformation

$$\theta' = \theta + \omega x_4, \quad x'_4 = x_4, \quad r' = r. \tag{16}$$

that is to say, the line-element

$$(S) \qquad ds^2 = (1 - r^2\omega^2)dx_4^2 - dr^2 - r^2 d\theta^2 - 2\omega r^2 d\theta dx_4.$$

In this case, taking r, θ as x_1, x_2 respectively, the non-vanishing $g_{\iota\kappa}$ are

$$g_{11} = -1, \; g_{22} = -r^2, \; g_{24} = -\omega r^2, \; g_{44} = 1 - \omega^2 r^2.$$

From these we derive, by (13), as the only surviving Christoffel symbols,

$$\begin{bmatrix} 12 \\ 2 \end{bmatrix} = -r, \begin{bmatrix} 12 \\ 4 \end{bmatrix} = -\omega r, \begin{bmatrix} 14 \\ 2 \end{bmatrix} = -\omega r, \begin{bmatrix} 14 \\ 4 \end{bmatrix} = -\omega^2 r,$$

$$\begin{bmatrix} 22 \\ 1 \end{bmatrix} = r, \; \begin{bmatrix} 24 \\ 1 \end{bmatrix} = \omega r, \begin{bmatrix} 44 \\ 1 \end{bmatrix} = \omega^2 r.$$

Next we have, the determinant of the form (S),

$$g = g_{11}(g_{22}\, g_{44} - g_{24}^2) = r^2,$$

and

$$g^{11} = -1, \; g^{22} = \frac{\omega^2 r^2 - 1}{r^2}, \; g^{24} = \omega, \; g^{44} = 1,$$

while all other $g^{\iota\kappa}$ vanish. Thus we find, by (14), as the only non-vanishing symbols,

$$\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} = \frac{1 - 2\omega^2 r^2}{r}, \; \begin{Bmatrix} 12 \\ 4 \end{Bmatrix} = -2\omega r, \begin{Bmatrix} 14 \\ 2 \end{Bmatrix} = \frac{\omega}{r}(1 - 2\omega^2 r^2),$$

$$\begin{Bmatrix} 14 \\ 4 \end{Bmatrix} = -2\omega^2 r, \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} = -r, \begin{Bmatrix} 44 \\ 1 \end{Bmatrix} = -\omega^2 r, \; \begin{Bmatrix} 24 \\ 1 \end{Bmatrix} = -\omega r,$$

again seven in number. Substituting these Christoffel symbols into (15), with $\iota = 1, 2, 4$ (for $r, \theta, x_4 = ct$), we have the equations of the world-geodesics, *i.e.*, the equations of motion of a free particle in the system S,

$$\left.\begin{aligned} \ddot{r} &= r\ (\dot{\theta}+\omega\dot{x}_4)^2 \\ \ddot{\theta} &= -\frac{2\dot{r}}{r}(1-2\omega^2r^2)\,(\dot{\theta}+\omega\dot{x}_4) \\ \ddot{x}_4 &= \quad 4\omega r\,.\,\dot{r}\ \ (\dot{\theta}+\omega\dot{x}_4), \end{aligned}\right\} \tag{17}$$

where the dots stand for derivatives with respect to s. In virtue of the identical equation $\dot{s}=1$, *i.e.*,

$$(1-r^2\omega^2)\dot{x}_4{}^2-\dot{r}^2-r^2\,\dot{\theta}^2-2\omega r^2\,\dot{\theta}\,\dot{x}_4=1, \tag{18}$$

one, say the third of (17), should be a consequence of the remaining two.* Thus, the proper equations of motion in the S-system being the first two alone, we can use (18) to eliminate from them $\dot{x}_4$, and to replace d/ds by d/dt.

In the first place, to see the approximate meaning of these equations of motion, consider the case of small velocities dr/dt, $rd\theta/dt$ (as compared with c), and of small values of ωr. [Notice that, by (16), $\bar{\omega}=\omega c$ is an angular velocity, in its dimensions at least, so that $\omega r=\bar{\omega}r/c$ is a pure number.] Thus $ds \doteq dx_4=cdt$, $\dot{x}_4 \doteq 1$, and the approximate equations of motion of a free particle in S are

$$\frac{dr^2}{dt^2} = r\left(\bar{\omega}+\frac{d\theta}{dt}\right)^2, \tag{a}$$

$$r\,\frac{d^2\theta}{dt^2} = -2\frac{dr}{dt}\left(\bar{\omega}+\frac{d\theta}{dt}\right).$$

In Cartesians, $x=r\cos\theta$, $y=r\sin\theta$, these equations are identical with

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \bar{\omega}^2x+2\,\bar{\omega}\,\frac{dy}{dt} \\ \frac{d^2y}{dt^2} &= \bar{\omega}y-2\,\bar{\omega}\,\frac{dx}{dt}\,. \end{aligned}\right\} \tag{b}$$

The reader will recognize at once in the right hand member of equation (a) or in the first terms of (b) the purely radial *centrifugal acceleration* (or 'force' per unit mass), provided,

*The verification may be left to the reader as an exercise.

of course, that he is at all willing to interpret $\tilde{\omega}$, in accordance with the transformation $\theta' = \theta + \tilde{\omega} t$, as the angular velocity of the system S (say, plane disc) relatively to the galilean S'. The second terms of (b) express then the *Coriolis acceleration*.

If we so desire we may, with Einstein, reckon these accelerations to the gravitational ones, especially if we are confined to the (rotating) system S. The centrifugal acceleration, at least, is radial, though away from the origin. The Coriolis acceleration, however, is perpendicular to the velocity and, therefore, generally oblique. Certainly we have in (17) a field of acceleration, but the only feature this has in common with a gravitational field is that all bodies placed in it will behave alike. But unlike gravitational fields they cannot be deduced from the distribution of matter. Yet Einstein would not like to have us distinguish them from gravitational fields. If so, then $\left\{ {22 \atop 1} \right\}$, $\left\{ {24 \atop 1} \right\}$, $\left\{ {44 \atop 1} \right\}$ contribute to the centrifugal, and $\left\{ {12 \atop 2} \right\}$, $\left\{ {14 \atop 2} \right\}$ to the Coriolis field. But until we are told how to derive these non-permanent 'fields' as gravitational effects of all the masses of the universe turning around S,* all this will be an idle question of pure nomenclature. We may leave it here for the present.

In the second place, returning to the rigorous equations (17), consider a particle, placed (by an S-inhabitant) at any point r_o, θ_o of the disc S and left there, at the instant $t=0$, to its own fate. If it is nailed down it will, of course, remain there for ever, being simply part of this reference system. But let it be a *free* particle from $t=0$ onwards. In short, let $\dot{r} = \dot{\theta} = 0$, for $t=0$. Then, by (17), we shall have, for that instant, $\ddot{\theta} = 0$ so that the particle will not evince any tendency of moving transversally, and

$$\frac{d^2r}{ds^2} = \frac{d}{dx_4}\left(\dot{x}_4 \frac{dr}{dt}\right) = \omega^2 r \dot{x}_4^2.$$

By (18), $\dot{x}_4^2 = (1 - r^2\omega^2)$, and since $\dot{r}_o = 0$, the last equation will become, rigorously, and always for $t=0$,

$$\frac{d^2r}{dt^2} = \tilde{\omega}^2 r.$$

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*This was tried by H. Thirring but not very successfully.

In fine, our particle will initially experience the familiar centrifugal acceleration.* It will fly off, for an S'-observer at a (straight) tangent, but from the S-standpoint at a spiral-shaped orbit.

This is perhaps the clearest way of stating the relation of our system S to the galilean S'. The reader need not, however, think of S at this stage as a material rigid disc rotating uniformly with respect to the fixed stars, although a uniform rotation is just one of the possible motions of a relativistically rigid body (Born, Herglotz). Notwithstanding that S was called, in passing, a disc, it will be safer to treat it here simply as a system derived from S' by the transformation (16) with $\tilde{\omega}$ as constant.

As to the orbit of a free particle relatively to S, its equation could be derived, not without some trouble, from the differential equations (17). This, however, can be done much easier by transforming the orbit from S' to S. In fact, the former being a galilean system, a free particle describes in it, uniformly, a straight line. Its equation can be written

$$r' \cos \theta' = r_o' = \text{const.},$$

where r_o' is the shortest distance of the straight orbit from the origin. Transformed by (16) the orbit in S will be

$$\frac{r_o}{r} = \cos\,(\theta + \tilde{\omega} t'),$$

and since $v't' = \sqrt{r^2 - r_o^2}$, where v' is the constant S'-velocity of the particle, we shall have ultimately, as the orbit of a free particle in S,

$$\frac{r_o}{r} = \cos\left[\theta \pm \frac{\tilde{\omega} r_o}{v'} \sqrt{\frac{r^2}{r_o^2} - 1}\,\right], \tag{19}$$

*One of Einstein's most vigorous exponents, de Sitter, sees herein a particularly extravagant property of the rotating system. Thus in Monthly Notices of the Roy. Astron. Soc., vol. 77 (1916), p. 176, de Sitter says: 'For $r\omega < 1$' [and, as we saw, for any $r\omega$] 'it is a physical impossibility for a material body to be at rest in the system B' [our S]. 'This shows the irreality of the coordinates', etc. But such is, in reality, the behaviour of free particles in a system rotating relatively to the stars, independently of any theory.

which is a kind of spiral. Notice in passing that between any two points A, B of the disc there are two such orbits, one leading from A to B and the other from B to A. Thus free motion in S is not reversible. This holds also for light rays, for which v' in (19) is to be given the value c. Light propagation is irreversible, and the two rays AB and BA enclose a certain area having the shape of a biconvex lens. But this by the way only.

The example of these two systems, S' and S, was here treated at some length in order to acquaint the reader with the handling of the geodesics and the Christoffel symbols. At the same time, however, it may serve as a good illustration of the purely formal part played by the principle of general relativity or general covariance. In fact, although the equations of motion of free particles have exactly the same form, (15) and (15) dashed, in the two systems, yet it is scarcely possible to imagine a more different phenomenal behaviour of free particles than is that in these two systems. The same remark applies to the light equations, $g_{\iota\kappa}' dx_\iota' dx_\kappa' = 0$ in S' and $g_{\iota\kappa} dx_\iota dx_\kappa = 0$ in S, exhibiting the same general form, but representing entirely different systems of optics; this difference goes even so far that, while in S' all light paths are reversible, in S, under appropriate conditions, Brown could see Jones without being visible to him, though both were well enough illuminated.

The purpose of these remarks is by no means to minimize the heuristic value of the general relativity principle, but only to show its purely formal nature. Notice that the case of the special relativity theory was altogether different; for, though giving privileges only to a certain class of systems, it claimed at least for all of them not only a formal equality, but an equal physical behaviour.

In passing from S' to S the Lorentz contraction was, for the sake of simplicity, altogether disregarded. This is the reason why the reader was warned not to take our S strictly as a rigid body rotating in S' but only as one obtained from S' by the simple mathematical transformation (16). Yet even with the said neglect the abstract S can at least

approximately stand for a rigid body, such as the earth (any plane r, θ parallel to its equator), endowed with a uniform spin relatively to the stars.

11. Leaving these simple examples let us once more return to the general equations of motion of a free particle,

$$\ddot{x}_\iota + \left\{ \begin{matrix} \alpha\beta \\ \iota \end{matrix} \right\} \dot{x}_\alpha \dot{x}_\beta = 0 \tag{15}$$

in order to see what form they assume when the $g_{\iota\kappa}$ differ but little from the galilean coefficients $\overline{g}_{\iota\kappa}$ and when $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ are small fractions, that is to say, when the velocity of the particle is small compared with that of light.

If the galilean line-element is written in Cartesians we have $\overline{g}_{11} = \overline{g}_{22} = \overline{g}_{33} = -1$, $\overline{g}_{44} = 1$, and

$$\left. \begin{array}{c} g_{11} = -1 + \gamma_{11}, \text{ etc.}, \; g_{44} = 1 + \gamma_{44}, \\ g_{\iota\kappa} = \gamma_{\iota\kappa}, \quad \kappa \neq \iota, \end{array} \right\} \tag{21}$$

where all the γ are small fractions of unity. With these values of the $g_{\iota\kappa}$ we could compute the approximate values of the Christoffel symbols appearing in (15), and thus arrive at the required equations. But it is simpler to return to $\delta \int ds = 0$, the original form of (15), to reduce the element ds and then to develop this form afresh.

Now, if $dx_1/cdt = \beta_1$, etc., $\beta_1^2 + \beta_2^2 + \beta_3^2 = \beta^2$, the line-element can be written

$$ds^2 = dx_4^2 \{ 1 - \beta^2 + \gamma_{44} - (\gamma_{11}\beta_1^2 + \ldots + \gamma_{33}\beta_3^2) + 2(\gamma_{12}\beta_1\beta_2 + \ldots$$
$$+ \gamma_{31}\beta_3\beta_1) + 2(\gamma_{14}\beta_1 + \ldots + \gamma_{34}\beta_3) \}.$$

All the squares and products of the β's are small of the second order. Thus, up to the third order we have

$$ds = L dx_4 \equiv dx_4 \sqrt{1 - \beta^2 + \gamma_{44} + 2(\beta_1 \gamma_{14} + \ldots + \beta_3 \gamma_{34})}, \tag{22}$$

and the equations of motion, $\int \delta L \, . \, dx_4 = 0$, will be

$$\frac{d}{dx_4}\left(\frac{\partial L}{\partial \beta_i}\right) - \frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2, 3.$$

Now,

$$\frac{\partial L}{\partial \beta_i} = \frac{1}{L}(\gamma_{4i} - \beta_i), \quad \frac{\partial L}{\partial x_i} = \frac{1}{L}\left[\frac{1}{2}\frac{\partial \gamma_{44}}{\partial x_i} + \beta_1 \frac{\partial \gamma_{41}}{\partial x_i} + \ldots\right],$$

and if the squares and the products of the γ's and their derivatives be neglected, we can put $L \doteq 1$ in the denominators. Thus the equations of motion will become

$$\frac{d}{dx_4}(\gamma_{4i} - \beta_i) = \tfrac{1}{2}\frac{\partial \gamma_{44}}{\partial x_i} + \beta_1 \frac{\partial \gamma_{41}}{\partial x_i} + \beta_2 \frac{\partial \gamma_{42}}{\partial x_i} + \beta_3 \frac{\partial \gamma_{43}}{\partial x_i},$$

or, developing the first term and remembering that the $\gamma_{\iota\kappa}$ differ from the $g_{\iota\kappa}$ only by additive constants,

$$\frac{d^2 x_i}{dt^2} = -\frac{c^2}{2}\frac{\partial g_{44}}{\partial x_i} + c\left[\frac{\partial g_{4i}}{\partial t} + \frac{dx_1}{dt}\left(\frac{\partial g_{4i}}{\partial x_1} - \frac{\partial g_{41}}{\partial x_i}\right) + \ldots \right.$$
$$\left. \ldots + \left(\frac{\partial g_{4i}}{\partial x_3} - \frac{\partial g_{43}}{\partial x_i}\right)\right]. \qquad (23)$$

These are Newton's equations of motion. The first terms on the right hand represent the rectangular components of an acceleration which is the gradient of a newtonian potential

$$\Omega = -\frac{c^2}{2} g_{44},$$

or, vice versa, $-g_{44}$ plays the rôle (apart from an additive constant) of the potential multiplied by $\frac{2}{c^2}$.

The second terms look less familiar. But their meaning can be made clear at once. They represent at any rate a certain acceleration field which need by no means be negligible in comparison with the newtonian one. The contributions of this field to the components of acceleration are

$$c\left[\frac{\partial g_{41}}{\partial t} + \frac{dx_2}{dt}\left(\frac{\partial g_{41}}{\partial x_2} - \frac{\partial g_{42}}{\partial x_1}\right) - \frac{dx_3}{dt}\left(\frac{\partial g_{43}}{\partial x_1} - \frac{\partial g_{41}}{\partial x_3}\right)\right], \text{ etc.,}$$

or in ordinary vector language, with $\mathbf{r} = (x_1, x_2, x_3)$ and $\mathbf{g}_4 = (g_{41}, g_{42}, g_{43})$

$$\frac{d^2\mathbf{r}}{dt^2} = c\frac{\partial \mathbf{g}_4}{\partial t} - c\mathrm{V}\frac{d\mathbf{r}}{dt}\operatorname{curl}\mathbf{g}_4.$$

This is manifestly the acceleration due to a *velocity field* $c\mathbf{g}_4$ impressed upon the system of reference. If this velocity field is homogeneous and constant in time, its contribution to acceleration is, of course, zero; but if it is heterogeneous and variable, it contributes to the acceleration of a free particle through its time rate of variation and through the vorticosity of its distribution. The simplest case occurs when $\mathbf{g}_4$ is a linear function of the coördinates alone, say

$$g_{41} = \frac{\tilde{\omega}}{c} x_2, \qquad g_{42} = -\frac{\tilde{\omega}}{c} x_1, \qquad g_{43} = 0,$$

where $\tilde{\omega}$ is a constant. Then c curl $\mathbf{g}_4$ is a (three-) vector of size $2\tilde{\omega}$ directed along the x_3-axis and the last equation gives

$$\frac{d^2x_1}{dt^2} = 2\tilde{\omega}\frac{dx_2}{dt}, \qquad \frac{d^2x_2}{dt^2} = -2\tilde{\omega}\frac{dx_1}{dt}, \qquad \frac{d^2x_3}{dt^2} = 0,$$

which is the *Coriolis acceleration* corresponding to a uniform rotation of the system with angular velocity $\tilde{\omega}$ round the x_3-axis (vectorially, with the angular velocity $\frac{c}{2}$. curl $\mathbf{g}_4$).

The reader will, perhaps, miss the centrifugal acceleration $\tilde{\omega}^2 r$, Coriolis' faithful companion. But this (having a scalar potential) is inseparable from g_{44}. It is included in g_{44} through the term $\frac{\tilde{\omega}r^2}{c^2}$, already familar to us from a previous example. The g_{4i} just given will be found by noticing that in (S), p. 30, $r^2 d\theta\, dx_4 = (x_1\, dx_2 - x_2\, dx_1)dx_4$. This settles the question.

In the more general case the spin $\frac{1}{2}c$. curl $\mathbf{g}_4$ will not be constant but will vary from point to point giving rise to a more complicated acceleration field.*

The approximate equations of motion (23) can now be written compactly, in three-dimensional vector language,

$$\frac{d^2\mathbf{r}}{dt^2} = -\nabla\frac{c^2 g_{44}}{2} + c\left[\frac{\partial \mathbf{g}_4}{\partial t} - \mathrm{V}\frac{d\mathbf{r}}{dt}\,\mathrm{curl}\,\mathbf{g}_4\right]. \qquad (23a)$$

*I propose to call so all fields corresponding to any ds^2, and to reserve the name of *gravitational* fields for those only which are 'permanent' or cannot be transformed away holonomously.

This equation brings at once into evidence the parts played by g_{44} and by the three g_{4i} condensed in $\mathbf{g}_4$. Both rôles may be equally conspicuous, and it would certainly be unjust to say, with Einstein, that it is only g_{44} which survives in this first approximation.

Einstein (*loc. cit.*, p. 817), in deriving the approximate newtonian equations from the rigorous ones, no doubt, through a too hasty computation of the Christoffel symbols, dropped altogether the second terms of our equations (23). And his 'slip' crept into the writings of de Sitter, Weyl and others. Einstein exclaims even (*ibid.*) in genuine surprise: 'The remarkable thing in this result $\left[\frac{d^2x_i}{dt^2} = -\frac{c^2}{2}\frac{\partial g_{44}}{\partial x_i}\right]$ is that only the component g_{44} of the fundamental tensor determines by itself, in a first approximation, the motion of a material particle'.

We shall return to these approximate equations of motion later on, after having set up Einstein's gravitational field-equations.

CHAPTER III.

Elements of Tensor Algebra and Analysis.

12. In order to be able to construct generally covariant laws or equations, such as Einstein's field-equations which will complete the fundamental part of his theory, some elementary notions of the Tensor Calculus are required. These I shall now proceed to give, without stopping to sketch the history of the origin and the growth of this powerful method of multidimensional analysis, which the reader will find in the preface to Ricci and Levi-Civita's paper on the Absolute Differential Calculus,* as the said branch of mathematics is called by these authors.

The relations and properties which are now to occupy our attention hold for a manifold of any number of dimensions. But, if not otherwise stated, we shall have in mind our *four*-dimensional world or space-time.

A world-point is given by four gaussian coördinates x_ι which, in general, are mere numbers or labels. As such they need not, as in the most familiar treatment, stand for such things as lengths or distances, or angles. By calling them 'labels' we do not mean, of course, that tetrads of numbers are being haphazardly, disorderly, attached to various events

*G. Ricci and T. Levi-Civita, *Méthodes de calcul differentiel absolu et leurs application*, Mathem. Annalen, vol. 54 (1900), pp. 125-201. A condensed account of this paper is given in J. E. Wright's *Invariants of Quadratic Differential Forms*, Cambridge Tracts, No. 9 (1908). Perhaps the easiest presentation of all that is required for relativistic applications is given in the second part (B.) of Einstein's own paper, *loc. cit.*, essentially reproduced in chap. III of A. S. Eddington's *Report*, Phys. Soc. London, 1918. Th subject is treated on original and very attractive lines by H. Weyl in *Raum, Zeit, Materie* (Springer, Berlin), 3rd ed., 1920. For geometrical applications the first volume of L. Bianchi's *Lezioni di Geometria Differenziale* (Spoerri, Pisa), 2 ed., 1902, can be most warmly recommended.

(world-points), but we assume that $x_1 = 7$, say, is a label attached to a whole connected three-dimensional continuum of world-points, and similarly for all other (real) numerical values of x_1. Likewise for the remaining coördinates, so that every world-point appears as the intersection of, or element common to, some four hypersurfaces of three dimensions. Manifestly, the use of such coördinates does not presuppose any idea of measurement. Again, in this abstract treatment of tensors as certain entities in the manifold, the question whether any one of the coördinates or its differential is space-like or time-like, is of no interest. It becomes relevant only when we come to apply these concepts to physical problems.

13. Such being the nature of the x_ι, pass from these to any other coördinates x_ι', through any holonomous transformation whatever, satisfying only the conditions of continuity, etc., as stated in chapter I. Then, as in (8a), the differentials dx_ι, *i.e.*, the coördinates of a world-point Q, a neighbour of $P(x_\iota)$, with P as origin, are transformed into

$$dx_\iota' = \frac{\partial x_\iota'}{\partial x_\kappa} dx_\kappa = \frac{\partial x_\iota'}{\partial x_1} dx_1 + \frac{\partial x_\iota'}{\partial x_2} dx_2 + \dots .$$

That is to say, the coördinates of Q with P as origin, are given, in the new system, by these linear homogeneous transformations of the old relative coördinates of the pair of points, with coefficients, $\partial x_\iota'/\partial x_\kappa$, which are some given functions of the position of P. Such an ordered point-pair, PQ, or the corresponding array of the dx_ι, is called a *vector*, in our case a *four-vector* or *world-vector*. From a more general standpoint to be explained presently its name is: a contravariant tensor of rank one.

Now, as in special relativity every tetrad which is transformed as the cartesian x, y, z and ct (*i.e.*, by the very special, linear, Lorentz transformation), so here the tetrad of infinitesimals dx_ι is made the prototype of all (contravariant) vectors. In other words, every tetrad of magnitudes A^ι which are transformed by the same rule as the dx_ι, *i.e.*,

$$A'^\iota = \frac{\partial x_\iota'}{\partial x_\kappa} A^\kappa, \tag{24}$$

is called a *contravariant vector* or a contravariant *tensor of rank one*, and A^1, A^2, etc., are called its *components*. (The upper position of the suffixes was proposed by Ricci and Levi-Civita and accepted by all authors. To be consequent one would have to write also dx^ι, as in fact is done by Weyl. But, for the sake of typographical convenience, an exception is being made for this prototype of all contravariant vectors.) It is scarcely necessary to say that, unlike the Cartesians in special relativity, the coördinates x_ι themselves do not form a vector; only their differentials do. In short, there are, in general, no finite position-vectors, but only differential ones. This, however, does not exclude the possibility of other finite vectors A^ι.

It is of particular importance to notice the *linearity and homogeneity* of the transformation formula (24) which will reappear in the case of all other tensors. The all-important consequence of this property is that if all components of a vector vanish in one system, they will vanish also in all other systems of coördinates. More briefly, if a vector A^κ vanishes in one system it will vanish also in any other system. Thus $A^\kappa=0$ will be a generally 'covariant' or, technically, contravariant law. This, of course, does not prejudice the question whether Nature is going to obey it.

Manifestly, if A^κ and B^κ are two contravariant vectors, so also are $A^\kappa+B^\kappa$ and $A^\kappa-B^\kappa$.

As dx_ι served as the standard of contravariant vectors, so do the operators (differentiators)

$$D_\iota = \frac{\partial}{\partial x_\iota}$$

serve as a prototype of another kind of vectors. We have, evidently,

$$D_\iota' = \frac{\partial x_\kappa}{\partial x_\iota'} D_\kappa,$$

and every tetrad of magnitudes B_ι which are transformed according to this rule,

$$B_\iota' = \frac{\partial x_\kappa}{\partial x_\iota'} B_\kappa, \tag{25}$$

is called a *covariant* vector or tensor of rank one. In comparison with (24), notice that the suffix of B' coincides with the lower (instead of upper) suffix in the coefficients. Although the prototype of these vectors consists of differentiators, the components B_ι of a covariant vector need not be operators, but may be magnitudes in the ordinary sense of the word. As in the previous case, $B_\kappa=0$ is a generally covariant equation or rather set of equations. And if B_κ and C_κ be two covariant vectors, so also are $B_\kappa \pm C_\kappa$. Needless to say that $A^\kappa + B_\kappa$ is neither a covariant nor a contravariant vector. In fact, it has no meaning if the system is not specified.

14. But, while the sum of a covariant and a contravariant vector is from the present point of view of no interest, the combination of their components

$$A^\iota B_\iota = A^1B_1 + A^2B_2 + A^3B_3 + A^4B_4,$$

which is called the *inner* or *scalar product*, has a very remarkable property. It is *invariant* with respect to any transformations of the coördinates. In fact, by (24) and (25),

$$A'^\iota B_\iota' = \frac{\partial x_\iota'}{\partial x_\kappa} A^\kappa \frac{\partial x_\lambda}{\partial x_\iota'} B_\lambda = \left(\frac{\partial x_\lambda}{\partial x_\iota'} \frac{\partial x_\iota'}{\partial x_\kappa}\right) A^\kappa B_\lambda.$$

But the x_1, x_2, etc., being mutually independent, the bracketed expression (to be summed over all ι) vanishes for all $\kappa \neq \lambda$ and equals 1 for $\kappa=\lambda$. Whence,

$$A'^\iota B_\iota' = A^\kappa B_\kappa = A^\iota B_\iota, \tag{26}$$

which was to be proved.

Any invariant, $S=S'$, is also called a *scalar* or a tensor of *rank zero*, since, in a manifold of n dimensions, it has n^0 components, *i.e.* but one component. Similarly, a vector or tensor of rank one, has $n^1=n$, in our case four, components. The question whether a scalar is a contravariant or a covariant tensor is idle. For it transforms into itself.

Vice versa, it can easily be proved that if B_κ be four (generally, n) magnitudes such that $A^\kappa B_\kappa$ is invariant for *any* contravariant A^κ, then B_κ is a covariant vector. And

the same thing is true if 'covariant' and 'contravariant' be exchanged with one another.

The product of a vector by a scalar is, obviously, again a vector of the same kind, and any number of vectors of the same kind multiplied by scalars and added together give again a vector of the same kind. Finally, notice that $A_\kappa B_\kappa$ and $A^\kappa B^\kappa$ are not invariant, and thus are no tensors at all.

15. As we just saw, the inner multiplication of a covariant and a contravariant vector degrades the rank of both factors giving a tensor of rank zero, a single component. Consider, on the other hand, what is known as *the outer product* of two vectors, of the same or of opposite kinds, *i.e.*, $A_\iota B_\kappa$, or $A^\iota B^\kappa$, or $A_\iota B^\kappa$. The suffixes being here different, no summation is understood, so that each of these symbols stands for $4^2=16$ (generally n^2) components. Let us take $A_\iota B_\kappa$ first, which is a short symbol for the array

$$\begin{matrix} A_1B_1 & A_1B_2 \ldots \\ A_2B_1 & A_2B_2 \ldots \\ \ldots & \ldots \end{matrix}$$

of sixteen magnitudes. Denote them by $M_{\iota\kappa}$ respectively. Their law of transformation is, by (25),

$$M'_{\iota\kappa} = \frac{\partial x_\alpha}{\partial x_\iota'}\frac{\partial x_\beta}{\partial x_\kappa'} M_{\alpha\beta}. \tag{27}$$

Every array of n^2 magnitudes $N_{\iota\kappa}$ (whether obtained by the outer multiplication of two covariant vectors or in any other way) which is transformed by the rule (27) is called a covariant tensor of *rank two*. It manifestly has again the property of vanishing in all systems, if it vanishes in one of them. In a four-manifold $N_{\iota\kappa}$ consists of 16 components.

In general $N_{\iota\kappa} \neq N_{\kappa\iota}$. If, in particular, $N_{\iota\kappa} = N_{\kappa\iota}$ the tensor is called *symmetrical*.

An example of such a tensor we had in $g_{\iota\kappa}$, called the fundamental tensor; cf. formula (9). Notice, however, that the tensor property of $g_{\iota\kappa}$ followed from the invariance of ds^2 which fixed the metrical properties of the world, whereas all our present considerations are entirely independent of the

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metrics of the manifold, and it is preferable to abstain from using them at this stage. Such properties as are impressed upon the general tensors by the metrics of the world will be treated in later sections.

In the meantime let us continue the *non-metrical* theory of tensors.

The symmetrical tensor $N_{\iota\kappa}$ consists in general of $\frac{1}{2}n(n+1)$, and for $n=4$, of ten different components. It can be easily proved, by (27), that its symmetry is an invariant property, *i.e.*, that if $N_{\iota\kappa}=N_{\kappa\iota}$ in one system, we have also $N'_{\iota\kappa}=N'_{\kappa\iota}$ in any other system. A covariant symmetrical tensor of rank two can be constructed at once from a covariant vector, to wit by forming its outer self-product, $A_{\mu\nu}=A_\mu A_\nu=A_\nu A_\mu=A_{\nu\mu}$.

If $N_{\iota\kappa}=-N_{\kappa\iota}$, for all ι, κ, we have an *antisymmetrical* (or *skew*) tensor. Since $N_{\kappa\kappa}=-N_{\kappa\kappa}$ means $N_{\kappa\kappa}=0$, a whole diagonal of components vanish, and thus only $\frac{1}{2}n(n+1)-n=\frac{1}{2}n(n-1)$ non-vanishing and independent components are left, the surviving ones being oppositely equal in pairs. Thus an antisymmetric tensor in a four-world consists of six independent components, and is therefore called a *six-vector*, in the present case a covariant six-vector. With such six-vectors the reader is already acquainted from the special relativistic treatment of the electromagnetic field. We shall see them at work in a similar duty in general relativity later on.

As the symmetry so also the antisymmetry is an invariant property, *i.e.*, $N_{\iota\kappa}=-N_{\kappa\iota}$ is transformed into $N'_{\iota\kappa}=-N'_{\kappa\iota}$.

Any tensor $N_{\iota\kappa}$ can be split at once into a symmetrical and an antisymmetrical one. For we have identically

$$N_{\iota\kappa}=\tfrac{1}{2}(N_{\iota\kappa}+N_{\kappa\iota})+\tfrac{1}{2}(N_{\iota\kappa}-N_{\kappa\iota}),$$

and the first term represents a symmetrical, the second an antisymmetrical tensor.

Similarly to (27), and starting from the special tensor $A^\iota B^\kappa$, any array of n^2 magnitudes which are transformed by the rule

$$N'^{\iota\kappa}=\frac{\partial x_\iota'}{\partial x_\alpha}\frac{\partial x_\kappa'}{\partial x_\beta}N^{\alpha\beta} \tag{27a}$$

is called a *contravariant* tensor of rank two. If $N^{\iota\kappa}=N^{\kappa\iota}$, it is a symmetrical, and if $N^{\iota\kappa}=-N^{\kappa\iota}$, an antisymmetrical tensor. (A tensor $N^{\iota\kappa}$ need not be the product of two contravariant vectors.)

Lastly (starting from $A_\iota B^\kappa$), any array of n^2 magnitudes N_ι^κ which are transformed by the mixed rule

$$N_\iota'^\kappa = \frac{\partial x_\beta}{\partial x_\iota'}\frac{\partial x_\kappa'}{\partial x_\alpha}N_\beta^\alpha \tag{27b}$$

is called a *mixed* tensor of rank two, covariant with respect to its lower suffix ι, and contravariant with respect to its upper suffix or index κ.* Special cases of symmetry and antisymmetry as before. A new feature, however, offered by the mixed tensor is this. With any N_ι^κ make $\iota=\kappa$, getting N_κ^κ and, by the usual convention, sum over all κ. In other words add up all the components of the chief diagonal (slanting down from left to right) of the mixed tensor. The result will be a single magnitude. Now, the important thing is that this magnitude is a general invariant. In fact, by (27*b*),

$$N_\kappa'^\kappa = \left(\frac{\partial x_\beta}{\partial x'_\kappa}\frac{\partial x_\kappa'}{\partial x_\alpha}\right)N_\beta^\alpha\,;$$

but (as mentioned before) the bracketed expression is zero for all $\alpha\neq\beta$ and one for $\alpha=\beta$. Thus

$$N_\kappa'^\kappa = N_\alpha^\alpha = N_\kappa^\kappa\,,$$

which proves the proposition.

Thus, equalling the upper and the lower index and summing over it degrades the mixed tensor by two ranks giving, in the present case, a tensor of rank zero or an invariant (scalar). In other words,

$$N_\kappa^\kappa = N$$

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*It seems inappropriate to call 'suffix' (from *sub*, under) an upper mark or sign. I propose ,therefore, to call such signs by the more general name *index*. Since all English writing authors accepted the 'three-*index* symbols' and the 'four-*index* symbols' (of Christoffel and Riemann), they will perhaps not object to calling ι, κ indices.

is an *invariant of the tensor* N_ι^κ. We shall see presently that this procedure of equalling an upper to a lower index, called *contraction* (German 'Verjüngung') can be applied, with equal success, to a mixed tensor of any rank whatever. Notice, however, that this process is not applicable in the case of (purely) covariant or contravariant tensors. Thus, for instance, $M_{\kappa\kappa}=M_{11}+M_{22}+\ldots$ is *not* invariant, as a glance on (27) will suffice to show. In short, the diagonal sum of $M_{\iota\kappa}$ has no intrinsic meaning. Similarly, in the case of a four-vector, say, $A_1+\ldots+A_4$ is not an invariant.

16. The next step, leading to tensors of rank three, and so on, is obvious. Generally, any system of n^r (in our world, 4^r) magnitudes $N_{\iota\kappa\ldots}^{\alpha\beta\ldots}$, with r_1 lower and r_2 upper indices, which are transformed by the rule

$$(N_{\iota\kappa\ldots}^{\alpha\beta\ldots})' = \frac{\partial x_i}{\partial x_\iota'}\,\frac{\partial x_k}{\partial x_\kappa'}\,\frac{\partial x_\alpha'}{\partial x_a}\,\frac{\partial x_\beta'}{\partial x_b}\,\ldots\,N_{ik\ldots}^{ab\ldots} \tag{28}$$

is called a *mixed tensor of rank* $r=r_1+r_2$, covariant with respect to its r_1 lower, and contravariant with respect to its r_2 upper indices. If all the components of such a tensor vanish in one system they will also vanish in any other system of coördinates. Any tensor, therefore, can be used for writing down generally covariant laws.* In particular, if $r_1=0$, the tensor (28) is contravariant, of rank r_2; and if $r_2=0$, covariant of rank r_1. The sum of any number of tensors of the same rank and kind, each multiplied by any scalar, is again a tensor of the same rank and kind, the numbers r_1, r_2 retaining their significance.

17. *Contraction.* This process, already illustrated on the simplest example, can now be generally explained.

Let α be any upper and ι any lower index† of a mixed tensor of any rank r whatever. Put $\alpha=\iota$ and sum over α. Then the result will be a *tensor of rank* $r-2$, with r_1-1 covariant and r_2-1 contravariant indices.

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*In the less technical sense of the word.

†The place of α among the upper, and of ι among the lower indices is irrelevant.

The proof follows at once from (28). For the process gives us in the coefficients of transformation a term

$$\frac{\partial x_a}{\partial x_\iota'} \frac{\partial x_\iota'}{\partial x_i}$$

which vanishes for all $a \neq i$ and equals one for $a = i$, thus reducing (28) to

$$(N^{\beta \cdot \cdot \cdot}_{\kappa \cdot \cdot \cdot})' = \frac{\partial x_b}{\partial x_\kappa'} \frac{\partial x_\beta'}{\partial x_k} \ldots . N^{b \cdot \cdot \cdot}_{k \cdot \cdot \cdot} ,$$

which proves the statement.

This process of contraction can obviously be applied again and again, degrading the tensor each time by *two ranks* until there will be no upper or no lower indices left. In fine, the mixed tensor can be degraded until it becomes purely covariant or purely contravariant or (if $r_1 = r_2$) until it is reduced to a scalar or invariant.

Thus, for example, the tensor $A^{\alpha\beta}_{\iota\kappa\lambda}$ of rank five gives rise to

$$A^{\alpha\beta}_{\alpha\kappa\lambda} = A^{1\beta}_{1\kappa\lambda} + A^{2\beta}_{2\kappa\lambda} + \ldots ,$$

which is denoted by $A^{\beta}_{\kappa\lambda}$, and this tensor of rank three gives rise to

$$A^{\kappa}_{\kappa\lambda} = A_\lambda$$

which is a (covariant) tensor of rank one or a vector.

Again (as an example of $r_1 = r_2$), the tensor $A^{\alpha\beta}_{\iota\kappa}$ of rank four gives by contraction A^{β}_{κ}, and this tensor of rank two gives

$$A^{\kappa}_{\kappa} = A ,$$

a scalar. We may as well write at once $A^{\iota\kappa}_{\iota\kappa} = A$, the meaning and the value of A being the same as before. This final invariant may be considered as a property of the original tensor $A^{\alpha\beta}_{\iota\kappa}$.

In general every such *half-and-half tensor* ($r_1 = r_2$) will have the final scalar (A) as its *intrinsic* invariant*. And, as far as I can see, this is its only intrinsic invariant.

**i.e.* an invariant of its own, independent of any extraneous form such as ds^2 (or any auxiliary tensor, such as $g_{\iota\kappa}$) determining the metrics of the manifold.

On the other hand a purely covariant or contravariant tensor or an unequally mixed one ($r_1 \neq r_2$) cannot be contracted to an invariant. It seems that it has no intrinsic invariant at all, that is to say, that there are no processes which would lead to an invariant combination of the components of the original tensor itself (without using other tensors).

18. *The inner multiplication*, already mentioned in connection with vectors, can now be considered as an outer multiplication followed by a contraction.

Consider two tensors, generally mixed, one of rank $r = r_1 + r_2$, the other of rank $s = s_1 + s_2$. Combine (by ordinary multiplication) each of the n^r components of the former with each of the n^s components of the latter. The n^{r+s} magnitudes thus obtained will be the components of a tensor of rank $r+s$ with $r_1 + s_1$ covariant and $r_2 + s_2$ contravariant indices. That the entity thus arising is a tensor follows at once from (28).

Thus the outer product of two vectors is a tensor of rank two, $A_\iota B_\kappa = M_{\iota\kappa}$, $A_\iota B^\kappa = M_\iota^\kappa$. Similarly $A_{\alpha\beta} B_{\iota\kappa}$ is a covariant tensor of rank four, $M_{\alpha\beta\iota\kappa}$, and $A_{\alpha\beta} B_\gamma^{\iota\kappa} = N_{\alpha\beta\gamma}^{\iota\kappa}$ is a mixed tensor of rank five, and so on.

The outer multiplication combined with contraction (when there are indices to contract) gives the inner product. Thus the inner product of A_ι and B^κ is

$$A_\kappa B^\kappa = M_\kappa^\kappa = M,$$

an invariant.* The inner product of A^κ and $B_{\alpha\beta}$ is their outer product $M_{\alpha\beta}^\kappa$ degraded by contraction, *i.e.*, $M_{\alpha\beta}^\beta = M_\alpha$, a covariant vector. The inner product of $A_{\alpha\beta}$ and $B^{\iota\kappa}$ is their outer product $A_{\alpha\beta} B^{\iota\kappa} = M_{\alpha\beta}^{\iota\kappa}$ degraded (to the utmost) by two contractions,

$$M_{\kappa\kappa}^{\kappa\kappa} = M,$$

i.e., a scalar or invariant. *Vice versa*, if $A_{\alpha\beta}$ be any array of n^2 magnitudes such that $A_{\alpha\beta} B^{\iota\kappa}$ is an invariant for *any* contravariant $B^{\iota\kappa}$, then $A_{\alpha\beta}$ is a covariant tensor of rank two. This criterion of tensor character, already mentioned in connection with $A_\iota B^\kappa$, can be easily proved by writing down the

*There is no inner product of A_ι , B_κ .

transformation formula of the given factor (tensor). And it can be extended to any rank and kind, no matter whether the inner product is a scalar or a tensor of any rank higher than zero.

As we already know, the differential operators $D_\iota = \partial/\partial x_\iota$ have the character of the components of a covariant tensor of rank one. Therefore, the 'product' of this tensor into a *scalar* or scalar-field $f = f(x_1, x_2 \ldots)$, that is to say, the result of operating with D_ι upon f, will again be a covariant tensor of rank one or a *covariant vector*,

$$\frac{\partial f}{\partial x_\iota} = A_\iota. \tag{29}$$

But we cannot go further than that. That is to say, an iterated application of the operation D_κ does not give a tensor. Thus $\partial^2 f/\partial x_\iota \partial x_\kappa$ is *not* a tensor. Nor do, in the more general case of any vector B_ι, the n^2 derivatives $D_\kappa B_\iota = \partial B_\iota/\partial x_\kappa$ constitute a tensor. The different behaviour of $D_\kappa B_\iota$ and of products of magnitude-tensors lies herein that the operational tensor D_κ acts also on the coefficients $\partial x_\kappa/\partial x_\iota'$ of the transformation formula of B_ι. In fact, we have

$$D_\kappa' B_\iota' = \frac{\partial x_\alpha}{\partial x_\kappa'} D_\alpha \left(\frac{\partial x_\beta}{\partial x_\iota'} B_\beta \right),$$

and* this is not the same thing as $\dfrac{\partial x_\alpha}{\partial x_\kappa'} \dfrac{\partial x_\beta}{\partial x_\iota'} D_\alpha B_\beta$. The same remark applies, a fortiori, to higher derivatives of scalars and of tensors of any rank.

In fine, the only tensor derivable by simple differentiation, unaided by other auxiliaries (cf. *infra*), is the covariant vector (29) yielded by a scalar. The vector or vector-field $\partial f/\partial x_\iota$ is called *the gradient* of f. In the case of space-time it consists of four components.

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*Unless the coordinate transformations are linear as in the special relativity theory.

19. *Tensor properties in a metrical manifold.* Having sufficiently acquainted ourselves with the properties of tensors in themselves, let us now consider them in relation to the fundamental quadratic form $ds^2 = g_{\iota\kappa}\, dx_\iota\, dx_\kappa$ which converts the hitherto amorphous world into a *metrical or riemannian* manifold.*

It is of the utmost importance to grasp well this distinction between a riemannian and a non-metrical manifold and to understand the true rôle of ds^2 in converting the latter into the former.

Let us place ourselves yet for a while upon the non-metrical standpoint. Of all the tensors described in the preceding sections let us confine our attention upon the prototype of all (contravariant) vectors, the infinitesimal position-vector dx_ι. Any such vector represents ultimately but an ordered pair of points, $O(x_\iota)$ the origin, and $A(x_\iota + dx_\iota)$ the end-point of the vector. Imagine a whole bundle of such infinitesimal vectors OA, OB, OC, etc., all emerging from the same world-point O as origin. Now, from the non-metrical point of view, all these vectors have (apart from their origin) nothing in common with one another. That is to say, if two of them, say OA and OB, are at all distinct from one another, and if their components dx_ι do not happen to be proportional to one another (in which case we can say that the vectors have a common 'direction'), there is in either of them nothing, no property, with respect to which they could be compared. They are, as it were, perfect strangers to one another. Similarly, if we call 'angle' a vector-pair $\alpha = OA, OB$, there is nothing to base upon a comparison of two non-overlapping covertical angles α and $\beta = OC, OD$. In short, neither vectors nor angles (or other derived entities) have 'sizes'. There is, in fact, in the manifold itself nothing which could fix the mere meaning of such a concept. Of two vectors OA, OB nothing more can

*The name 'riemannian' manifold or n-space is being often used in this connection in view of the historical fact that Riemann was the first to base the general geometry of an n-space upon its line-element given by such a differential form, although Gauss was his great predecessor in the case of surface theory.

be said than that they are either identical (or co-directional, collinear) with or distinct from one another. The origin being the same,* the points A, B are either identical or distinct, and no other significant statement can be made about their relation.

But while there is nothing in the manifold itself to base a comparison of distinct infinitesimal vectors upon, we are at liberty to provide for it at our will if we so desire. This is done by introducing a standard or fundamental entity such as the quadratic form called the line-element. In other words, we surround the world-point $O(x_\iota)$ by a hypersurface, a three-dimensional (generally an $n-1$ dimensional) quadric and declare all vectors emerging from O and ending in any point P $(x_\iota+dx_\iota)$ of this surface to be equal in *size* or in *absolute value*, or in 'length', the usual name in the case of our three-space. It is precisely this metrical surface† which is expressed by

$$g_{\iota\kappa}\,dx_\iota\,dx_\kappa = ds^2 = \text{const.},$$

the numerical value of ds being the 'size' common to all these infinitesimal vectors or point-pairs.‡ The part played by this quadratic form is essentially the same as that of Cayley's 'absolute' or standard quadric (a real quadric leading to lobatchevskyan or hyperbolic, an imaginary quadric leading to elliptic, and the intermediate degenerate quadric leading to euclidean geometry), the only important difference being that Riemann's treatment is much more general. It covers

*We have limited the discussion to coinitial vectors solely for the sake of simplicity. All our remarks apply *a fortiori* to distant, non-coinitial bundles of vectors.

†The German geometers call it Eichfläche.

‡In Riemann's own treatment this rôle of the fundamental form impressed upon the manifold extends into distance, over all the manifold. That is to say, if $O'(y_\iota)$ be any other point and if a quadric $g_{\iota\kappa}dy_\iota dy_\kappa=\text{const.}$ be drawn around it with the same value of the constant as before, all the vectors of the bundle O' terminating upon this quadric are again said to have the same size as those of the bundle O. In this respect a somewhat more general standpoint was recently proposed by Weyl, in connection with his ideas on electromagnetism.

all metrical spaces (in technical language, of variable and anisotropic curvature), whereas Cayley's device gives us only a space of constant isotropic curvature, negative, zero, or positive. This fully corresponds to his starting point, which was that of projective geometry. Yet, and this is of particular interest in the present connection, Cayley recognized thoroughly the true rôle of all such standard entities. In fact, he tells us plainly that geometrical figures have no metrical properties in themselves. Their metrical properties such as those of the foci of a conic, etc., arise only by relating them to other figures, as the 'absolute' conic in the plane, or quadric in three-space.

The kind of metrics thus impressed upon a continuous manifold being essentially arbitrary, the utility of the metrical manifold thus obtained will, of course, from the physicist's standpoint, depend upon the interpretation which is given to the said 'size' of a position-vector, and to special lines of that metrical manifold, such as the geodesics, in terms of measuring rods, clocks, moving particles or light phenomena, and so on.

But without dwelling here any further upon such questions of a *concrete representation* let us turn to consider the purely mathematical consequences of the introduction of $g_{\iota\kappa}\, dx_\iota\, dx_\kappa$ as a fundamental differential form fixing the metrics of the manifold.

20. As in Cayley's case the geometrical figures in relation to his 'absolute', so here the tensors acquire some new properties in relation to the fundamental form or better, to its coefficients $g_{\iota\kappa}$. In fact, what determines the form are these coefficients, and we may look upon the matter in the following way.

Instead of declaring the fundamental quadratic form at the outset as an invariant, let us better say that the symmetrical array of 16 (generally n^2) magnitudes $g_{\iota\kappa}$ is being introduced as a *fundamental tensor*, symmetrical, of rank two and of the covariant kind, as defined in the preceding sections.

Combined with this fundamental tensor all other tensors of the previously amorphous manifold will acquire some

new properties. These and only these will now be their *metrical* properties.

To begin with the prototype of contravariant vectors, the infinitesimal vector dx_ι has had thus far no invariant of his own. But it will acquire one with the aid of the fundamental tensor. In fact, dx_ι being contravariant, denote it for the moment by X^ι. Form the outer product

$$g_{\alpha\beta}\, X^\iota X^\kappa$$

which will be a mixed tensor $A^{\iota\kappa}_{\alpha\beta}$. Contract it with respect to ι, α, getting $A^{\iota\kappa}_{\iota\beta} = A^\kappa_\beta$. Contract this again. Then the result will be $A^\kappa_\kappa = A$, a scalar or invariant. Or perform both contractions at once, and write ds^2 for A, returning to the original notation, thus

$$g_{\iota\kappa}\, dx_\iota\, dx_\kappa \equiv ds^2 = \text{invariant.}$$

In short, the inner product of the tensor $dx_\alpha\, dx_\beta$ into the fundamental tensor $g_{\iota\kappa}$ is an invariant. There is no objection to calling it *the invariant of* dx_ι as a short name for its *metrical or associated invariant.* Thus, thanks to $g_{\iota\kappa}$, the vector dx_ι has acquired an invariant. And it can now be compared through it with other vectors, no matter what their components. The value of ds^2 may be called *the norm*, and the absolute value of $\sqrt{\pm ds^2}$ *the size* of the vector dx_ι. Thus we can speak of two vectors dx_ι and dy_ι being equal in size, or one having twice the size of the other, and so on. In application to the four-world, a vector dx_ι of *no* size will be a light vector, a vector of negative norm a space-like, and one of positive norm a time-like vector.

Similarly, any other contravariant vector A^ι will have the metrical invariant

$$g_{\iota\kappa}\, A^\iota A^\kappa = A^2, \text{ say.*} \qquad (30)$$

*Of course, even in the amorphous manifold an invariant could be built up from A^ι by the aid of any covariant tensor $N_{\iota\kappa}$, but the choice of $N_{\iota\kappa}$ being entirely free, such an invariant would not have a fixed value. We fix it by introducing once for all a special tensor $g_{\iota\kappa}$ to serve for all other tensors.

In much the same way, if B_ι be any covariant vector, we shall have in

$$g^{\iota\kappa} B_\iota B_\kappa = B^2 \tag{30a}$$

an invariant, the norm of B_ι.

From a more general point of view we may call A^2, in (30), the tensor, of rank zero, *metrically associated* to A^ι, similarly, in (30a), B^2 to B_ι.

Moreover, we can easily construct associated tensors of a rank other than zero, and differing also in kind from the original tensor. Thus, to dwell still upon vectors,

$$g_{\iota\kappa} A^\kappa = A_\iota \tag{31}$$

will be the covariant vector metrically associated with the contravariant vector A^κ. We may call A_ι shortly *the conjugate* of A^ι. Similarly, starting from a covariant vector A_ι, we shall have the contravariant vector

$$g^{\kappa\iota} A_\kappa = A^\iota \tag{31a}$$

conjugate to A_ι.

Two questions naturally suggest themselves: Will the conjugate of the conjugate be the original vector? Have two conjugate vectors the same size or the same norm?

In order to answer these questions as well as for the sake of what will follow, let us first note a simple property of the tensors $g_{\iota\kappa}$ and $g^{\iota\kappa}$.. By definition, chap. II, $g^{\iota\kappa}$ is the minor of the determinant $g = |g_{\iota\kappa}|$, corresponding to its ι, κ-th element, divided by g itself. But g is equal to the sum of the products of the elements of its first column, say, into the corresponding minors, *i.e.*, $g = g_{a1}\, g g^{a1}$, whence $g_{a1}\, g^{a1} = 1$. Similarly for any other column (or row). Thus, underlining the index over which an expression is *not* to be summed,

$$g_{\mu\underline{\nu}}\, g^{\mu\underline{\nu}} = 1.$$

This is valid for every ν. Thus $g_{\mu\nu} g^{\mu\nu}$, summed over both indices, has the value 4 for our world, and n for an n-fold. Again, taking two different columns (or rows) of g, we shall easily prove that

$$g_{\mu\nu}\, g^{\mu\lambda} = 0.$$

Both properties can be united in a single formula

$$g_{\kappa\alpha}\, g^{\kappa\beta} = g_\alpha^\beta = \delta_\alpha^\beta\,, \tag{32}$$

where δ_α^β is the conventional symbol for 1 or 0 according as $\alpha=\beta$ or $\alpha\neq\beta$. This symbol is itself a mixed tensor.

We are now able to answer our two questions. First, the conjugate of the conjugate of the vector A_ι is, by the definitions (31), (31*a*),

$$g_{\alpha\kappa}\, g^{\kappa\iota}\, A_\iota = g_\alpha^\iota\, A_\iota = \delta_\alpha^\iota\, A_\iota = A_\iota\,,$$

i.e., the original vector. Similarly if we started with A^κ. Thus, *the conjugate of the conjugate is the original vector.*

Second, if A^ι be the conjugate of A_ι we have for the norm of the former vector, by (30) and (31*a*),

$$g_{\iota\kappa}\, A^\iota A^\kappa = g_{\iota\kappa}\, g^{\alpha\iota}\, A_\alpha\, g^{\beta\kappa} A_\beta = \delta_\kappa^\alpha\, A_\alpha\, g^{\beta\kappa} A_\beta \;=\; g^{\beta\kappa} A_\beta\, A_\kappa.$$

Thus any two conjugate vectors have *equal norms.*

The norm of A_ι and of A^ι can also be written $A_\iota A^\iota$, for this is again equal to $g_{\iota\kappa}\, A^\iota A^\kappa$. Thus, for instance, if $d\xi_\iota$ be the conjugate of the contravariant vector dx_ι, their common norm or the squared line-element can be written

$$ds^2 = dx_\iota\, d\xi_\iota\,. \tag{33}$$

21. In much the same way we can treat the metrical properties of tensors of any higher rank. To explain the method it will be enough to take up in some detail the second rank tensor $\mathrm{A}_{\iota\kappa}$. Its conjugate or supplement (Ergänzung) will be the contravariant tensor defined by

$$g^{\alpha\iota}\, g^{\beta\kappa}\, A_{\alpha\beta} = A^{\iota\kappa}\,,\ \text{or also}\ g_{\alpha\iota}\, g_{\beta\kappa}\, A^{\alpha\beta} = A_{\iota\kappa}\,. \tag{34}$$

The tensor $g^{\iota\kappa}$ itself is easily proved to be the supplement of the tensor $\mathrm{g}_{\iota\kappa}$.

The scalar or invariant of $A_{\iota\kappa}$ will be

$$g^{\iota\kappa}\, A_{\iota\kappa} = A_\kappa^\kappa = A\,. \tag{35}$$

A single contraction of $g^{\iota\kappa}\, A_{\alpha\beta}$ will give

$$g^{\iota\kappa}\, A_{\alpha\kappa} = A_\alpha^\iota\,,$$

a mixed tensor metrically associated with the covariant $A_{\alpha\kappa}$.

The supplement of the supplement (or the conjugate of the conjugate) is again the original tensor, for

$$g_{\alpha\iota}\, g_{\beta\kappa}\, A^{\iota\kappa} = g_{\alpha\iota}\, g_{\beta\kappa}\, g^{\gamma\iota}\, g^{\delta\kappa}\, A_{\gamma\delta} = \delta_{\alpha}^{\gamma}\, \delta_{\beta}^{\delta}\, A_{\gamma\delta} = A_{\gamma\delta}\,.$$

The tensors $A_{\iota\kappa}$ and $A^{\iota\kappa}$ have the same scalar A, (35). In fact, the scalar of $A^{\iota\kappa}$ is

$$g_{\iota\kappa}\, A^{\iota\kappa} = g_{\iota\kappa}\, g^{\alpha\iota}\, g^{\beta\kappa} A_{\alpha\beta} = \delta_{\kappa}^{\alpha}\, g^{\beta\kappa}\, A_{\alpha\beta} = g^{\beta\kappa} A_{\beta\kappa} = A\,.$$

Since $g^{\mu\nu} A_{\mu\nu}$ is an invariant, $B_{\iota\kappa} = g_{\iota\kappa}\, g^{\mu\nu} A_{\mu\nu}$ is again a tensor; Einstein calls it the *reduced* tensor belonging to $A_{\mu\nu}$.

Notice that neither a covariant nor a contravariant tensor has an invariant independent of the metrical tensor; only a mixed tensor, B_{ι}^{κ}, has such an invariant, to wit $B = B_{\lambda}^{\lambda}$. This is a privilege of mixed tensors of even rank with $r_1 = r_2$, and of these tensors only.

The investigation of other metrical properties of tensors of the second and higher ranks may be left to the reader. Exercises of such a kind will soon make him familiar with this broad and powerful algorithm.

22. *Angle and volume.* Consider any two coinitial infinitesimal vectors dx_{ι}, dy_{ι}. These are contravariant vectors. Therefore, as we already know, the inner product

$$g_{\iota\kappa}\, dx_{\iota}\, dy_{\kappa}$$

will be an invariant. It will remain invariant when divided by the sizes of both vectors. By an obvious generalisation of the familiar cosine formula this invariant is used to define the *angle* ϵ made by the two vectors, thus

$$\cos \epsilon = \frac{g_{\iota\kappa}\, dx_{\iota}\, dy_{\kappa}}{ds\, d\sigma}\,, \tag{36}$$

where $ds^2 = g_{\iota\kappa}\, dx_{\iota}\, dx_{\kappa}$, $d\sigma^2 = g_{\iota\kappa}\, dy_{\iota}\, dy_{\kappa}$, The two vectors are said to be *orthogonal* or perpendicular upon one another if

$$g_{\iota\kappa}\, dx_{\iota}\, dy_{\kappa} = 0\,.$$

Generally, the angle between any two vectors A^{ι}, B^{ι}, whose norms as defined by (30) are A^2 and B^2, will be determined by

$$\cos \epsilon = \frac{g_{\iota\kappa} A^\iota B^\kappa}{AB} , \tag{37}$$

and the vectors will be orthogonal if $g_{\iota\kappa} A^\iota A^\kappa = 0$. Similarly for covariant vectors, with the only difference that $g_{\iota\kappa}$ is replaced by $g^{\iota\kappa}$. Let A_ι, B_ι be the conjugates of A^ι, B^ι; then

$$g^{\iota\kappa} A_\iota B_\kappa = g^{\iota\kappa} g_{\alpha\iota} g_{\beta\kappa} A^\alpha B^\beta = \delta^\kappa_\alpha g_{\beta\kappa} A^\alpha B^\beta = g_{\alpha\beta} A^\alpha B^\beta,$$

and since A_ι, B_ι have the same norms as A^ι, B^ι, we see that the angle between the conjugates is the same as between the original vectors.

The integral $\int dx_1\, dx_2 \ldots dx_n$ extended over a domain of the manifold is, by a well-known theorem, transformed into $\int J dx_1'\, dx_2' \ldots dx_n'$, where J is the Jacobian $\left| \frac{\partial x_\iota}{\partial x_\kappa'} \right|$, as in (7). On the other hand the determinant g of the fundamental tensor (called also the discriminant of the fundamental quadratic form) is transformed into

$$g' = \left| g'_{\iota\kappa} \right| = \left| \frac{\partial x_\alpha}{\partial x_\iota'} \frac{\partial x_\beta}{\partial x_\kappa'} g_{\iota\kappa} \right| = \left| \frac{\partial x_\mu}{\partial x_\nu'} \right|^2 \cdot \left| g_{\iota\kappa} \right| ,$$

the last step being based on the multiplication rule of determinants. Thus

$$g' = J^2 g. \tag{38}$$

Consequently, the integral

$$\int \sqrt{g}\ dx_1\, dx_2 \ldots dx_n \tag{39}$$

is a scalar or an *invariant* of the n-dimensional domain of integration.

In the case of the four-dimensional world the determinant g is always negative.* Thus the invariant expression

*In a galilean domain and in Cartesians $g = -1$, by (1b), p. 6. By (38), therefore, it is also negative, always for a galilean domain, in any other system of coördinates derived from the Cartesians by a holonomous transformation. Now, although a non-galilean domain cannot be made galilean by a holonomous transformation, yet we know that in all practical cases the $g_{\iota\kappa}$ differ but very little from the galilean coefficients. Thus g will also in general be negative.

$$d\Omega = \sqrt{-g}\ dx_1\, dx_2\, dx_3\, dx_4 \qquad (40)$$

will be real. This is taken as 'the local measure' of the size or *volume* of an infinitesimal world-domain. For in the local (cartesian) coördinates u_ι, for which $g=-1$, this expression becomes $du_1\, du_2\, du_3\, du_4 = c\, dt\, dx\, dy\, dz$. The latter product is called by Einstein 'the natural' volume-element. Apart from names, the important thing to notice is the general invariance of the expression (40) as such or when integrated over any world-domain.

Consider any sub-domain of the world, of three, two or one dimension. This can be represented by expressing the x_ι as functions of three, two or one parameter respectively. The differentials dx_ι will be homogeneous linear functions of the differentials dp_a of these independent parameters. Thus the line-element within the sub-domain will be of the form

$$ds^2 = h_{\alpha\beta}\, dp_\alpha\ dp_\beta\ ,\ h_{\alpha\beta} = h_{\beta\alpha}\ ,$$

and the sub-domain, therefore, will again be a metrical manifold (a three-space, surface or line) in Riemann's sense of the word, and if $h = |h_{\alpha\beta}|$,

$$d\Omega = \sqrt{h}\ dp_1\, dp_2 \ldots$$

will (apart perhaps from a factor $\sqrt{-1}$) again be an invariant measure of an element (volume, area, length) of the sub-domain.

Thus, in the case of a one-dimensional sub-domain or line,

$$dx_\iota = \frac{dx_\iota}{dp}\ dp,$$

and
$$ds^2 = g_{\iota\kappa}\, \frac{dx_\iota}{dp} \cdot \frac{dx_\kappa}{dp}\ dp^2 = h_{11}\, dp^2, \text{ say.}$$

In this case $h = h_{11}$ and, therefore,

$$d\Omega = \sqrt{h_{11}}\ dp,$$

which is ds itself, as it should be.

For a two-dimensional sub-domain or surface we have

$$ds^2 = h_{11}\, dp_1^2 + 2h_{12}\, dp_1\, dp_2 + h_{22}\, dp_2^2,$$

where
$$h_{ab} = g_{\iota\kappa}\ \frac{\partial x_\iota}{\partial p_a}\ \frac{\partial x_\kappa}{\partial p_b}\ .$$

Thus,

$$d\Omega = \sqrt{h}\ dp_1\, dp_2,$$

where

$$h = g_{\iota\kappa}\ \frac{\partial x_\iota}{\partial p_1}\ \frac{\partial x_\kappa}{\partial p_1} \cdot g_{\iota\kappa}\ \frac{\partial x_\iota}{\partial p_2}\frac{\partial x_\kappa}{\partial p_2} - \left(g_{\iota\kappa}\ \frac{\partial x_\iota}{\partial p_1}\ \frac{\partial x_\kappa}{\partial p_2}\right)^2 .$$

—4

23. *Differentiation based on metrics.* We have already seen (p. 49) that if f be a scalar or invariant, $\partial f/\partial x_\iota$, the gradient of f, is a covariant vector. This is independent of the metrics of the manifold. But, as was then pointed out, the iterated application of the operation $\partial/\partial x_\iota$ would not lead to tensors; nor would its application to a vector A_ι or another tensor yield by itself, unaided by auxiliaries such as $g_{\iota\kappa}$, a tensor. But the introduction of the metrical tensor opens in this respect new and important possibilities.

It was remarked by Christoffel as long ago as 1869 that if A_ι be a covariant tensor, so is

$$A_{\iota\kappa} = \frac{\partial A_\iota}{\partial x_\kappa} - \left\{ \begin{matrix} \iota\kappa \\ \alpha \end{matrix} \right\} A_\alpha, \tag{41}$$

namely covariant, of rank two. Similarly if $B_{\iota\kappa}$ be a covariant tensor of rank two,

$$B_{\iota\kappa\lambda} = \frac{\partial B_{\iota\kappa}}{\partial x_\lambda} - \left\{ \begin{matrix} \iota\lambda \\ \alpha \end{matrix} \right\} B_{\alpha\kappa} - \left\{ \begin{matrix} \kappa\lambda \\ \alpha \end{matrix} \right\} B_{\iota\alpha} \tag{42}$$

is again a covariant tensor of rank three; similarly

$$B^\kappa_{\iota\lambda} = \frac{\partial B^\kappa_\iota}{\partial x_\lambda} - \left\{ \begin{matrix} \iota\lambda \\ \alpha \end{matrix} \right\} B^\kappa_\alpha + \left\{ \begin{matrix} \lambda\alpha \\ \kappa \end{matrix} \right\} B^\alpha_\iota \tag{42a}$$

is a mixed tensor of rank three, and so on. But it will be enough to consider here at some length the first case (41) only, especially as the other cases can be derived from it. The operation indicated in (41) is called *covariant differentiation*, and its result $A_{\iota\kappa}$ *the covariant derivative* or *the expansion* (Erweiterung) of A_ι.

If B^ι be a contravariant vector,

$$B^{\iota\kappa} = g^{\alpha\kappa} \left[\frac{\partial B^\iota}{\partial x_\alpha} + \left\{ \begin{matrix} \alpha\beta \\ \iota \end{matrix} \right\} B^\beta \right] \tag{41a}$$

is a contravariant tensor of rank two, *the contravariant derivative* of B^ι. But for our purposes it will suffice to consider only the covariant differentiation.

That (41) represents a covariant tensor can be proved in a variety of ways. The most instructive of these is perhaps

that given by Einstein, since it makes immediate use of the equations of geodesics, and the rôle of the Christoffel symbols* appearing in (41) is thus far known to us only in connection with these world-lines. Einstein's reasoning is as follows:

Let f be a scalar or better a scalar field (*i.e.* an invariant function of position within the world). Differentiate it twice along any world-line. Then

$$\frac{d^2f}{ds^2} = \frac{\partial f}{\partial x_\iota}\frac{d^2x_\iota}{ds^2} + \frac{\partial^2 f}{\partial x_\iota \partial x_\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}$$

will again be an invariant. Let the line be a geodesic. Then $\frac{d^2x_\iota}{ds^2} = -\left\{\begin{matrix}\alpha\beta\\ \iota\end{matrix}\right\}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}$, and the invariant will assume the form

$$\frac{d^2f}{ds^2} = \left[\frac{\partial^2 f}{\partial x_\alpha \partial x_\beta} - \left\{\begin{matrix}\alpha\beta\\ \iota\end{matrix}\right\}\frac{\partial f}{\partial x_\iota}\right]\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}.$$

Since the contravariant tensor (of rank two) $dx_\alpha dx_\beta$ is arbitrary (for from a given point a geodesic can be drawn in any direction, *i.e.* with arbitrary ratios of dx_1, dx_2, etc.) and its product into the bracketed term is invariant, the latter, *i.e.*

$$f_{\iota\kappa} \equiv \frac{\partial^2 f}{\partial x_\iota \partial x_\kappa} - \left\{\begin{matrix}\iota\kappa\\ \alpha\end{matrix}\right\}\frac{\partial f}{\partial x_\alpha} \tag{43}$$

is a covariant tensor of rank two. This proves the proposition for the special vector $A_\iota = \partial f/\partial x_\iota$. To prove it for any covariant vector, notice that any such vector A_ι can be represented by the sum of four (generally n) terms of the form $\psi \partial f/\partial x_\iota$, where ψ and f are scalars. Thus it is enough to prove that

$$\frac{\partial}{\partial x_\kappa}\left(\psi\frac{\partial f}{\partial x_\iota}\right) - \left\{\begin{matrix}\iota\kappa\\ \alpha\end{matrix}\right\}\psi\frac{\partial f}{\partial x_\alpha}$$

is a tensor. But this is equal to

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*Notice in passing that $\left\{\begin{matrix}\iota\kappa\\ \lambda\end{matrix}\right\}$ is *not* a tensor.

$$\psi f_{\iota\kappa} + \frac{\partial f}{\partial x_\iota} \frac{\partial \psi}{\partial x_\kappa}$$

which, being the sum of covariant tensors of rank two, is itself a tensor of the same kind and rank.

Thus the tensor character of the derivative (41) of any vector A_ι is proved. Notice that for constant $g_{\iota\kappa}$ (galilean world) the Christoffel symbols vanish and this covariant tensor of derivatives reduces to an array of ordinary derivatives $\partial A_\iota/\partial x_\kappa$.

The proof of the tensor character of (42), which can be easily deduced from that of (41), may be left to the care of the reader. It is interesting to note that the covariant derivative of the metrical tensor $g_{\iota\kappa}$ itself vanishes dentically. In fact, substituting in (42) $g_{\iota\kappa}$ for $B_{\iota\kappa}$ we have

$$g_{\iota\kappa\lambda} = \frac{\partial g_{\iota\kappa}}{\partial x_\lambda} - \begin{Bmatrix} \iota\lambda \\ \alpha \end{Bmatrix} g_{\alpha\kappa} - \begin{Bmatrix} \kappa\lambda \\ \alpha \end{Bmatrix} g_{\alpha\iota},$$

and since

$$g_{\alpha\kappa} \begin{Bmatrix} \iota\lambda \\ \alpha \end{Bmatrix} = g_{\alpha\kappa}\, g^{\alpha\beta} \begin{bmatrix} \iota\lambda \\ \beta \end{bmatrix} = \delta^\beta_\kappa \begin{bmatrix} \iota\lambda \\ \beta \end{bmatrix} = \begin{bmatrix} \iota\lambda \\ \kappa \end{bmatrix}, \tag{43}$$

which will also be useful in other connections, and similarly for the last term, we have

$$g_{\iota\kappa\lambda} = \frac{\partial g_{\iota\kappa}}{\partial x_\lambda} - \begin{bmatrix} \iota\lambda \\ \kappa \end{bmatrix} - \begin{bmatrix} \kappa\lambda \\ \iota \end{bmatrix}.$$

But by the definition (13) of the symbols, and since $g_{\iota\kappa} = g_{\kappa\iota}$, we have

$$\begin{bmatrix} \kappa\lambda \\ \iota \end{bmatrix} + \begin{bmatrix} \iota\lambda \\ \kappa \end{bmatrix} = \frac{\partial g_{\iota\kappa}}{\partial x_\lambda}. \tag{44}$$

Thus, $g_{\iota\kappa\lambda} = 0$, identically.

Let $A_{\iota\kappa}$ be as in (41), where A_ι stands for any covariant vector. Then, since the second term in (41) is symmetrical in ι, κ, $\partial A_\iota/\partial x_\kappa - \partial A_\kappa/\partial x_\iota = A_{\iota\kappa} - A_{\kappa\iota}$, being the difference of two tensors, is again a tensor. This tensor is called *the rotation* of the covariant vector A_ι, and can be written

$$\text{Rot}\,(A_\iota) = \frac{\partial A_\alpha}{\partial x_\beta} - \frac{\partial A_\beta}{\partial x_\alpha}. \tag{45}$$

This covariant tensor of rank two is manifestly antisymmetrical, *i.e.*, in the case of a four-manifold, a *six-vector*. Notice that although the proof of the tensor character of the rotation was based on the metrical formula (41), yet the rotation itself, as defined by (45), is entirely independent of the metrical properties impressed upon the manifold. It contains no trace of the metrical tensor $g_{\iota\kappa}$.

The same is true of a tensor of rank three which can be deduced from (42). Let in that formula $B_{\iota\kappa}$ be an *antisymmetric* tensor or six-vector. Then

$$B_{\iota\kappa\lambda}+B_{\kappa\lambda\iota}+B_{\lambda\iota\kappa}=\frac{\partial B_{\iota\kappa}}{\partial x_\lambda}+\frac{\partial B_{\kappa\lambda}}{\partial x_\iota}+\frac{\partial B_{\lambda\iota}}{\partial x_\kappa}. \tag{46}$$

Thus the right hand member is again a tensor. This is called the antisymmetric *expansion of the six-vector* $B_{\iota\kappa}$. It will, together with the rotation (45), be of use in connection with electromagnetism.

Another tensor derived from a six-vector of equal importance in the said connection is

$$A^\iota=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\kappa}(\sqrt{-g}\,A^{\iota\kappa})\equiv \mathrm{Div}\,(A^{\iota\kappa}), \tag{47}$$

a contravariant vector, called *the divergence of the contravariant six-vector* $A^{\iota\kappa}=-A^{\kappa\iota}$. The proof of its tensor character, to be based on (42), can be omitted here.

Finally let us mention, without proof, that

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\kappa}(\sqrt{-g}\,A^\kappa)\equiv \mathrm{div}\,(A^\kappa), \tag{48}$$

called *the divergence of the contravariant vector* A^κ, is a scalar or invariant.

24. *The Riemann-Christoffel tensor* is of such capital importance for Einstein's gravitation theory, and for the geometry of any riemannian manifold, as to deserve to be treated at some length.

It is a metrical tensor of rank four, built up of $g_{\iota\kappa}$ and their first and second derivatives, known to the general geometers since the time of Riemann.

It expresses the so-called curvature properties of a manifold or n-space whose metrical relations are fixed by the tensor $g_{\iota\kappa}$, and to Einstein it served as the material for building up his gravitational field-equations.

In order to arrive at this all-important tensor let us start from an arbitrary covariant vector A_ι and let us write down its *second covariant derivative*, that is to say the covariant derivative of the tensor $A_{\iota\kappa}$ which is the covariant derivative of A_ι, *i.e.*, by (42),

$$A_{\iota\kappa\lambda} = \frac{\partial A_{\iota\kappa}}{\partial x_\lambda} - \begin{Bmatrix} \iota\lambda \\ \alpha \end{Bmatrix} A_{\alpha\kappa} - \begin{Bmatrix} \kappa\lambda \\ \alpha \end{Bmatrix} A_{\iota\alpha},$$

where $A_{\iota\kappa}$ is as in (41). Similarly, transposing κ and λ, let us write the second covariant derivative

$$A_{\iota\lambda\kappa} = \frac{\partial A_{\iota\lambda}}{\partial x_\kappa} - \begin{Bmatrix} \iota\kappa \\ \alpha \end{Bmatrix} A_{\alpha\lambda} - \begin{Bmatrix} \lambda\kappa \\ \alpha \end{Bmatrix} A_{\iota\alpha}.$$

Either being a third-rank tensor, so will be their difference

$$A_{\iota\kappa\lambda} - A_{\iota\lambda\kappa} = \frac{\partial A_{\iota\kappa}}{\partial x_\lambda} - \frac{\partial A_{\iota\lambda}}{\partial x_\kappa} - \begin{Bmatrix} \iota\lambda \\ \alpha \end{Bmatrix} A_{\alpha\kappa} + \begin{Bmatrix} \iota\kappa \\ \alpha \end{Bmatrix} A_{\alpha\lambda}.$$

This is, by (41),

$$\left[\frac{\partial}{\partial x_\kappa}\begin{Bmatrix} \iota\lambda \\ \alpha \end{Bmatrix} - \frac{\partial}{\partial x_\lambda}\begin{Bmatrix} \iota\kappa \\ \alpha \end{Bmatrix}\right] A_\alpha + \left[\begin{Bmatrix} \iota\lambda \\ \alpha \end{Bmatrix}\begin{Bmatrix} \alpha\kappa \\ \beta \end{Bmatrix} - \begin{Bmatrix} \iota\kappa \\ \alpha \end{Bmatrix}\begin{Bmatrix} \alpha\lambda \\ \beta \end{Bmatrix}\right] A_\beta.$$

In the second term the indices α and β over which the sum is to be taken can be interchanged. Thus $A_{\iota\kappa\lambda} - A_{\iota\lambda\kappa}$ is the inner product of an arbitrary covariant vector A_α into the sum of the two bracketed expressions. This sum, therefore,

$$B^\alpha_{\iota\kappa\lambda} = \frac{\partial}{\partial x_\kappa}\begin{Bmatrix} \iota\lambda \\ \alpha \end{Bmatrix} - \frac{\partial}{\partial x_\lambda}\begin{Bmatrix} \iota\kappa \\ \alpha \end{Bmatrix} + \begin{Bmatrix} \iota\lambda \\ \beta \end{Bmatrix}\begin{Bmatrix} \beta\kappa \\ \alpha \end{Bmatrix} - \begin{Bmatrix} \iota\kappa \\ \beta \end{Bmatrix}\begin{Bmatrix} \beta\lambda \\ \alpha \end{Bmatrix}, \tag{49}$$

is a mixed tensor of rank four. This is *the Riemann-Christoffel tensor* which, for reasons to appear presently, may as well be called *the curvature tensor*.

Strictly speaking, Riemann's own system of *four-index symbols* $(\iota\mu,\ \lambda\kappa)$, discovered in 1861 in connection with a problem in heat conduction (*Mathematische Werke*, 2nd ed., p. 391), is the purely covariant tensor associated with (49), to wit

$$B_{\iota\mu\lambda\kappa} = (\iota\mu,\ \lambda\kappa) = g_{\mu\alpha}\, B^{\alpha}_{\iota\kappa\lambda}\,. \tag{50}$$

From this we have conversely,

$$B^{\alpha}_{\iota\kappa\lambda} = g^{\mu\alpha}(\iota\mu,\ \lambda\kappa). \tag{50a}$$

Also the latter tensor was used in geometry for a long time.*

The Riemann symbols are, for an n-space, n^4 in number, and for our world, therefore, as many as 256. But they are bound to one another by the linear relations

$$(\iota\mu,\ \kappa\lambda) = -(\mu\iota,\ \kappa\lambda),\ (\iota\mu,\ \kappa\lambda) = -(\iota\mu,\ \lambda\kappa),\ (\iota\mu,\ \kappa\lambda) = (\kappa\lambda,\ \iota\mu),$$

$$(\iota\mu,\ \kappa\lambda) + (\iota\lambda,\ \mu\kappa) + (\iota\kappa,\ \lambda\mu) = 0,$$

so that the number of essentially different, *i.e.* linearly independent symbols is reduced to

$$N = \frac{n^2(n^2-1)}{12}\,. \tag{51}$$

For a proof see, for instance, Killing, *loc. cit.*, p. 228.

In the case of a one-dimensional manifold, a line, there is no such non-vanishing symbol. In fact, although a line may be 'curved' from the standpoint of two- or more-dimensional beings in whose space it is imbedded, yet it has no intrinsic properties of its own to distinguish it from other lines, nor one of its parts from another. Take, for instance, a plane curve. If $\Delta\omega$ be the angle between the tangents at two points separated by the arc Δs, the curvature of the line is defined as the limit $d\omega/ds$. Now, this curvature is often called an intrinsic property of the line, because (unlike the sloping of the line) it is independent of a coördinate system laid in

*Cf. for instance L. Bianchi, 1902, *loc. cit.*, p. 72, where it is denoted by $\{\iota\alpha,\ \lambda\kappa\}$. The geometrical applications of the Riemann symbols are fully treated in vol. I of Bianchi's work. See also W. Killing's *Nicht-Euklidische Raumformen*, Leipzig (Teubner), 1885.

that plane, yet it is entirely meaningless if the line is not conceived as a sub-domain of the plane. For so is the angle $\Delta\omega$. And from the bidimensional standpoint every curve is developable upon every other.

In the case of a surface, $n=2$, there is, by (51), essentially just one Riemann symbol, namely

$$(12,\ 12),$$

(21, 21) being equal, and (12, 21), (21, 12) oppositely equal to it, and all others being zero. This unique symbol divided by the discriminant g is a differential invariant of the surface (or of its metrical form $ds^2=g_{11}dx_1^2+2g_{12}dx_1dx_2+g_{22}dx_2^2$). This invariant,

$$K=\frac{(12,\ 12)}{g}=\frac{(12,\ 12)}{g_{11}g_{22}-g_{12}^2}, \tag{52}$$

is the familiar gaussian curvature of the surface, its reciprocal being the product of the two principal radii of curvature. This *is* an intrinsic metrical property of the surface, requiring for its general definition or its numerical evaluation no reference whatever to a third dimension. In fact, (52) contains only the metrical tensor components $g_{\iota\kappa}$ and their first and second derivatives with respect to any gaussian coördinate system spread over the surface itself as a network of lines. The curvature thus defined, in general variable from point to point, can be evaluated at any spot by dividing the excess of the angle sum (over a straight angle) of an infinitesimal triangle by the area of the triangle. Again, as is well known, the necessary and sufficient condition for a surface to be developable upon a euclidean plane or for its fundamental form to be holonomously transformable into

$$ds^2=dx^2+dy^2$$

is the vanishing of K, *i.e.* of (12, 12), and herewith the vanishing of the whole tensor of Riemann symbols.

For a three-space there are, by (51), *six*, and for the world or space-time as many as *twenty* independent Riemann symbols. A five-space has fifty independent symbols, and so on. But, no matter what the number of dimensions, the

Riemann symbols always represent the curvature relations of the manifold, and their vanishing continues to form the condition of an important property of the metrical form of the manifold.

To begin with the latter, suppose all $g_{\iota\kappa}$ are constant over a domain of the world. Then all $(\iota\mu, \lambda\kappa)$, and therefore also all the components of the tensor $B^{\alpha}_{\iota\kappa\lambda}$ vanish throughout the domain. This then is *the necessary* condition for a domain of the world to be *galilean*, *i.e.*, for the line-element to be holonomously transformable into $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. It was proved by Lipschitz that this, *i.e.*

$$B^{\alpha}_{\iota\kappa\lambda} = 0,$$

is also *the sufficient* condition for the said reducibility (to a form with constant coefficients).

In the second place, concerning the curvature relations, consider a surface σ or a two-dimensional sub-domain of the world, or of any metrical manifold. More especially, let σ be a geodesic surface. This can be defined as follows. From a point $O(x_\iota)$ draw two infinitesimal vectors $d\xi_\iota$, $d\eta_\iota$ and consider the pencil of infinitesimal vectors

$$dx_\iota = \alpha d\xi_\iota + \beta d\eta_\iota\,,$$

where α, β are free coefficients. Draw from O the geodesic lines with each of these vectors as initial direction. The surface thus constructed will be a *geodesic surface* through O. Its normal ν will be defined by the infinitesimal vector dy_ι orthogonal to $d\xi_\iota$ and $d\eta_\iota$, *i.e.*, such that

$$g_{\iota\kappa}\, dy_\iota d\xi_\kappa = g_{\iota\kappa}\, dy_\iota d\eta_\kappa = 0,$$

and therefore also $g_{\iota\kappa}\, dy_\iota dx_\kappa = 0$. The geodesic surface $\sigma = \sigma_\nu$ will thus be completely determined by O, one of its points, and by its orientation, given by the normal just explained. The line-element of the manifold (world) will give for the line-element of this surface, at O, an expression of the form

$$ds^2 = h_{11}\, dp_1^2 + 2h_{12}\, dp_1\, dp_2 + h_{22}\, dp_2^2,$$

and the gaussian curvature of σ_ν will be, as before, with $h = h_{11}\, h_{22} - h_{12}^2$,

$$K_\nu = \frac{(12,\ 12)_h}{h}. \tag{52a}$$

This is, according to Riemann's definition, *the curvature of the manifold, at O, corresponding to the orientation ν of the geodesic surface.* The suffix h has to remind us that the Riemann symbol is to be formed with $h_{\iota\kappa}$ as the fundamental tensor (of the sub-manifold σ_ν). It remains to express (52a) in terms of the original tensor $g_{\iota\kappa}$ of the manifold and the vectors $d\xi_\iota$, $d\eta_\iota$ defining the orientation of the surface. This gives ultimately*

$$K_\nu = \frac{1}{h} \Sigma' \, (\iota\lambda,\ \kappa\mu)_g \begin{vmatrix} d\xi_\iota & d\eta_\iota \\ d\xi_\lambda & d\eta_\lambda \end{vmatrix} \cdot \begin{vmatrix} d\xi & d\eta_\kappa \\ d\xi_\mu & d\eta_\mu \end{vmatrix}, \tag{53}$$

where

$$h = \Sigma' \begin{vmatrix} g_{\iota\kappa} & g_{\iota\mu} \\ g_{\lambda\kappa} & g_{\lambda\mu} \end{vmatrix} \cdot \begin{vmatrix} d\xi_\iota & d\eta_\iota \\ d\xi_\lambda & d\eta_\lambda \end{vmatrix} \cdot \begin{vmatrix} d\xi_\kappa & d\eta_\kappa \\ d\xi_\mu & d\eta_\mu \end{vmatrix},$$

the dashed sums to be extended only over such combinations of the indices for which $\iota < \lambda$ and at the same time $\kappa < \mu$.

This will suffice to show the rôle of the four-index symbols in determining the riemannian curvature of a metrical manifold of any number of dimensions. In general, the curvature will not only vary from point to point but will also be different for different surface orientations (ν). In short, the manifold will in general be heterogeneous and anisotropic with regard to its curvature. Such, for instance, will be the world as the seat of a permanent gravitational field. On the other hand, a galilean domain, for which all $(\iota\lambda,\ \kappa\mu)$ vanish, will have everywhere and for every orientation the curvature zero. In other words, it will be *flat* or *homaloidal*. The next simple case is that of a manifold of positive or negative *constant curvature*, for which, that is, $K_\nu = K$ is constant and equal for all directions of ν. It may be interesting to note that the necessary and sufficient condition for the constancy and isotropy of curvature is

*Cf. Bianchi, *loc. cit.*, pp. 340-343.

$$(\iota\lambda,\ \kappa\mu) = K(g_{\iota\kappa}\, g_{\lambda\mu} - g_{\iota\mu}\, g_{\lambda\kappa}), \tag{54}$$

for all values of the indices ι, κ, λ, μ. These are partial differential equations of the second order for the $g_{\iota\kappa}$ with K as a constant coefficient. They exhibit the flat manifold, $K=0$, as a sub-case.

Other concepts connected with the system of riemannian curvatures K_ν, such as the mean curvature, will be given later on in connection with gravitational problems. Here our purpose was only to show that the curvature of the four-world and, in fact, of a metrical manifold of any number of dimensions, is a concept as definite and essentially as simple as that of an ordinary surface. The only difference is that of a possible anisotropy of K_ν for three- and more dimensional manifolds, whereas there is, of course, no such possibility in the case of a surface.

We are now ready to explain the use made by Einstein of the Riemann-Christoffel or *the curvature-tensor* in constructing his gravitational field equations. These will occupy our attention in the next chapter.

CHAPTER IV.

The Gravitational Field-equations, and the Tensor of Matter.

25. As was pointed out on several occasions, the fundamental, metrical tensor $g_{\iota\kappa}$ of the world determines, through the line-element $ds^2 = g_{\iota\kappa}\, dx_\iota\, dx_\kappa$, its null-lines and its geodesics, and these, in virtue of the explained concrete representation, rule the propagation of light and the motion of a free particle, respectively. It remains to build up generally covariant laws or equations which would enable us to determine the metrical tensor $g_{\iota\kappa}$ itself. Needless to say that in looking after such equations Einstein had in view, from the outset, the gravitational field. Of this he knew that to a certain degree of approximation it was represented by the (non-covariant) differential equation of Laplace-Poisson for the ordinary potential,

$$\nabla^2 \Omega = -4\pi\rho,$$

the gradient of the potential Ω giving the right hand member of Newton's (approximately valid) equations of motion. The equation of Laplace-Poisson being of the second order it was natural to look for a tensor containing the *second* derivatives of the metrical tensor components together with the $g_{\iota\kappa}$ themselves and their first derivatives.

Such a tensor lay ready in the treasury of the geometry of n-dimensional spaces since the time of Riemann, and it represented, moreover, certain intrinsic properties of any such metrical manifold, its curvature properties. This was the covariant tensor of the four-index symbols $(\iota\mu, \lambda\kappa)$ or the associated mixed Riemann-Christoffel tensor $B^{a}_{\iota\kappa\lambda}$. It was natural, therefore, and Einstein himself relates to us that such was his first thought, to utilize for the purpose in hand this very tensor.

As we saw in the last chapter, the vanishing of this tensor expressed a simple and at the same time a profound feature of a metrical manifold, to wit, the $\frac{1}{12} n^2(n^2-1)$ independent equations

$$B^{\alpha}_{\iota\kappa\lambda} = 0$$

formed the sufficient and necessary condition for the flatness of the manifold. In our case twenty such equations form the necessary and sufficient condition for a world-domain to be essentially galilean, *i.e.*, for its line-element to be holonomously transformable into $c^2dt^2-dx^2-dy^2-dz^2$.

A domain, therefore, in which there is no gravitational field, *i.e.*, no premanent field of acceleration, will certainly be characterized by the generally covariant equations $B^{\alpha}_{\iota\kappa\lambda}=0$. The same equations might at first suggest themselves for the description of a gravitational field outside of matter. It will be seen, however, after a moment's reflection that they would be too stringent for such purposes. In fact, the field of acceleration surrounding the sun, say, can certainly not be transformed away holonomously. The said equations would thus be too stringent for such fields and, in fact, for any acceleration field which, in our nomenclature, is a *permanent* field, *i.e.*, not to be got rid of by any holonomous transformations of the coördinates. At the same time the $g_{\iota\kappa}$ to be determined are only *ten* in number, forming a symmetrical tensor of rank two, while the Riemann-Christoffel tensor is of rank four, and consists of twenty independent components. Such considerations led Einstein to require for the gravitational field outside of matter a set of broader equations, yet of the second order, and ten in number. For this purpose the symmetrical tensor derived from $B^{\alpha}_{\iota\kappa\lambda}$ by *contraction* with respect to α, λ naturally suggested itself.

In fine, writing $B^{\alpha}_{\iota\kappa\alpha}=G_{\iota\kappa}$, Einstein's *field equations outside of matter* are

$$G_{\iota\kappa} = 0. \qquad \text{(III}^\circ\text{)}$$

That $G_{\iota\kappa}$, obtained by contraction from the mixed tensor of rank four, is itself a covariant tensor of rank two, we know

from the preceding chapter. Moreover, by (49), to be contracted with respect to λ, α, we have

$$G_{\iota\kappa} = \frac{\partial}{\partial x_\kappa}\begin{Bmatrix} \iota\alpha \\ \alpha \end{Bmatrix} - \frac{\partial}{\partial x_\alpha}\begin{Bmatrix} \iota\kappa \\ \alpha \end{Bmatrix} + \begin{Bmatrix} \iota\alpha \\ \beta \end{Bmatrix}\begin{Bmatrix} \beta\kappa \\ \alpha \end{Bmatrix} - \begin{Bmatrix} \iota\kappa \\ \beta \end{Bmatrix}\begin{Bmatrix} \beta\alpha \\ \alpha \end{Bmatrix}, \tag{55}$$

or, after some simple transformations which can here be omitted,

$$\left.\begin{aligned} G_{\iota\kappa} &= \begin{Bmatrix} \iota\alpha \\ \beta \end{Bmatrix}\begin{Bmatrix} \kappa\beta \\ \alpha \end{Bmatrix} - \frac{\partial}{\partial x_\alpha}\begin{Bmatrix} \iota\kappa \\ \alpha \end{Bmatrix} + S_{\iota\kappa}, \\ S_{\iota\kappa} &= \frac{\partial^2 \log\sqrt{-g}}{\partial x_\iota\,\partial x_\kappa} - \begin{Bmatrix} \iota\kappa \\ \alpha \end{Bmatrix}\frac{\partial \log\sqrt{-g}}{\partial x_\alpha}. \end{aligned}\right\} \tag{55a}$$

Now, $\begin{Bmatrix} \iota\kappa \\ \alpha \end{Bmatrix} = \begin{Bmatrix} \kappa\iota \\ \alpha \end{Bmatrix}$, so that $S_{\iota\kappa}$ (which is not a tensor) is symmetrical, and such being also the first two terms in (55a), $G_{\iota\kappa}$ is seen to be a *symmetrical* covariant tensor, of rank two; $G_{\iota\kappa} = G_{\kappa\iota}$.

Thus Einstein's field equations (III°), valid outside of matter, are ten in number, and such is exactly the number of the metrical tensor components $g_{\iota\kappa}$. The field equations would then give us a system of ten differential equations of the second order for ten unknown functions $g_{\iota\kappa}$ of the coördinates. As a matter of fact, however, there exist between the covariant derivatives $G_{\iota\kappa\lambda}$ of the $G_{\iota\kappa}$ and the derivatives $\partial G/\partial x_\iota$ of the invariant $G = g^{\iota\kappa} G_{\iota\kappa}$ four identical relations (based upon certain identical differential relations discovered by Bianchi), to wit

$$G_\iota \equiv g^{\kappa\lambda} G_{\iota\kappa\lambda} = \tfrac{1}{2}\frac{\partial G}{\partial x_\iota}, \quad \iota = 1,\ 2,\ 3,\ 4. \tag{56}$$

Owing to these four identities, to which we shall have to return later on, only six of the field equations are mutually independent, leaving therefore four of the $g_{\iota\kappa}$ or any four functions of the $g_{\iota\kappa}$ free or undetermined. Such, however, should from the general relativistic standpoint be the case.

In fact, from this point of view one would expect beforehand the field equations or any differential laws to be such as to leave us a perfectly free choice of the system of coordiantes. Einstein himself, for instance, makes use of this freedom by putting in most of his formulae $\sqrt{-g}=1$, which, by (55*a*), reduces his field equations to

$$G_{\iota\kappa} = -\frac{\partial}{\partial x_\alpha}\left\{ {\iota\kappa \atop \alpha} \right\} + \left\{ {\iota\alpha \atop \beta} \right\}\left\{ {\kappa\beta \atop \alpha} \right\} = 0, \qquad (57)$$

and leaves him still a threefold freedom of choice. The latter can often be used with advantage by making $g_{14}=g_{24}=g_{34}=0$. It will be kept in mind, however, that the equations, such as (57), thus simplified do not retain their form under general transformations. They are only useful as technical devices offering some advanatges in the treatment of special problems. The generally covariant form of the field equations is only that obtained by equating to zero the complete or general value of $G_{\iota\kappa}$, such as (55) or (55*a*).

26. In order to see the relation of Einstein's field equations to the more familiar Laplace equation, let us evaluate the curvature tensor $G_{\iota\kappa}$ for the case of a 'weak' field, *i.e.* differing but litle from a galilean domain.*

Thus, using a quasi-cartesian system of coördinates, let the fundamental tensor differ but little from the galilean tensor $\bar{g}_{\iota\kappa}$, *i.e.*, as in (21), let

$$g_{\iota\kappa} = \bar{g}_{\iota\kappa} + \gamma_{\iota\kappa} ,$$

where all the $\gamma_{\iota\kappa}$ are small fractions. Then the products of the Christoffel symbols in (55) will be small of the second order, and the tensor in question will be reduced to

$$G_{\iota\kappa} = \frac{\partial}{\partial x_\kappa}\left\{ {\iota\alpha \atop \alpha} \right\} - \frac{\partial}{\partial x_\alpha}\left\{ {\iota\kappa \atop \alpha} \right\}.$$

Here, up to second order terms,

$$\left\{ {\iota\kappa \atop \alpha} \right\} = \bar{g}^{\alpha\beta}\left[{\iota\kappa \atop \beta} \right] = \bar{g}^{\alpha\alpha}\left[{\iota\kappa \atop \alpha} \right],$$

*Notice in passing that all gravitational fields known from experience are 'weak' in this sense of the word.

and since $\bar{g}^{11} = \bar{g}^{22} = \bar{g}_{33} = -1$, $\bar{g}_{44} = 1$, while all other $\bar{g}^{\iota\kappa}$ vanish,

$$\left\{ {\iota\kappa \atop i} \right\} = -\left[{\iota\kappa \atop i} \right], \quad i=1, 2, 3;$$

$$\left\{ {\iota\kappa \atop 4} \right\} = \left[{\iota\kappa \atop 4} \right].$$

Thus, using the index i for 1, 2, 3 and summing every term in which i occurs twice over 1, 2, 3, we have the approximate curvature tensor

$$G_{\iota\kappa} = \frac{\partial}{\partial x_i}\left[{\iota\kappa \atop i} \right] - \frac{\partial}{\partial x_4}\left[{\iota\kappa \atop 4} \right] + \frac{\partial}{\partial x_\kappa}\left(\left[{\iota 4 \atop 4} \right] - \left[{\iota i \atop i} \right] \right). \tag{58}$$

In the present connection the only interesting component is that corresponding to $\iota = \kappa = 4$. This is, by (58), and on substituting the values (13) for the Christoffel symbols,

$$G_{44} = -\tfrac{1}{2}\nabla^2 g_{44} + \frac{\partial^2 g_{4i}}{\partial x_i \partial x_4} - \frac{1}{2}\frac{\partial^2 g_{ii}}{\partial x_4^2}, \tag{58a}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x_i^2}$ is the well-known Laplacian $\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$.

If the field is stationary, the second and the third terms vanish and Einstein's last field equation, $G_{44} = 0$, reduces to the familiar *equation of Laplace*

$$\nabla^2 g_{44} = 0. \tag{59}$$

At the same time, as we saw before (p. 36), the equations of motion assume, in absence of g_{4i}, the form of Newton's equations

$$\frac{d^2 x_i}{dt^2} = \frac{\partial \Omega}{\partial x_i}, \tag{23b}$$

where the potential $\Omega = -\dfrac{c^2}{2} g_{44}$, differing only by a constant factor from g_{44}, again satisfies Laplace's equation. The complete contents of Newton's law of gravitation, thus far outside of matter, appear as a first approximation to Einstein's field equations and his equations of motion of a free particle.

27. The ten field equations $G_{\iota\kappa}=0$ are valid outside of 'matter', *i.e.*, as is expressly stated by Einstein, in such domains of space-time in which there is not only no matter in the ordinary sense of the word but also no electromagnetic field, or, in fact, no distribution of energy of any origin other than gravitational. Following Einstein's example the word 'matter' will be used to cover all such cases. This will harmonise with the property of energy already familiar to us from special relativity,* namely of possessing inertia, an amount of energy U being equivalent to an inert mass U/c^2, which, by the law of proportionality, is also its heavy or gravitational mass.

As we saw before, the rôle of the newtonian gravitation potential Ω is, in a first approximation, taken over by the tensor component g_{44} multiplied by $-c^2/_2$. The vanishing of G_{44} was approximately equivalent to Laplace's equation $\nabla^2\Omega=0$ which holds outside of matter. Within matter Laplace's equation is replaced in the classical theory of gravitation by the more general equation of Laplace-Poisson,

$$\nabla^2\Omega=-4\pi\rho,$$

where ρ is the density of mass in astronomical units.† Now, since G_{44} reduces approximately, in a stationary field, to $-\frac{1}{2}\nabla^2 g_{44} \doteq \dfrac{1}{c^2}\nabla^2\Omega$, the idea easily suggests itself to make

$$G_{44} \doteq -\frac{4\pi\rho}{c^2}, \tag{60}$$

and to consider this as the equation or at least as one of the field equations within matter. But, needless to say, such a single equation would not by itself serve any relativistic purpose. What is required is a system of ten equations, of

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*And partly even from pre-relativistic considerations, such as in Mosengeil's investigations on an enclosure filled with radiation or those made in connection with Poynting's light-pressure experiments.

†It will be kept in mind that a mass m in astronomical units is defined by $\dfrac{m^2}{r^2}=$force, so that its dimensions are

$$[m]=[\text{length}\times(\text{velocity})^2].$$

which this should be one. In other words, the tensor $G_{\iota\kappa}$ has to be made equal or proportional to a symmetrical co-variant tensor of rank two somehow associated with 'matter' and having for its 44-component the density ρ or what approximately reduces to the usual mass density and therefore, apart from a constant factor, to energy density. Now, such a tensor was familiar from the special relativity theory under the name of *stress-energy tensor* often abbreviated to *energy tensor*. The merit of having introduced this concept into modern physics is chiefly due to Minkowski and Laue, preceded in non-relativistic physics by Max Abraham. The energy tensor made its first appearance in electromagnetism, in connection with the ponderomotive properties of an electromagnetic field,* as the symmetrical array or matrix

$$\mathfrak{S} = \begin{vmatrix} f_{11} & f_{12} & f_{13} & p_1 \\ f_{21} & f_{22} & f_{23} & p_2 \\ f_{31} & f_{32} & f_{33} & p_3 \\ p_1 & p_2 & p_3 & -u \end{vmatrix} = \begin{vmatrix} f & \mathbf{p} \\ \mathbf{p} & -u \end{vmatrix}$$

consisting of the six components $f_{ik} = f_{ki}$ of the maxwellian electromagnetic stress, of twice the three components p_i of electromagnetic momentum (or Poynting's energy flux) and of the density u of electromagnetic energy. The physical significance of this tensor or matrix was that its product into the operational matrix

$$\mathrm{lor} = \left| \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t} \right|$$

gave the ponderomotive force $\mathbf{P}$ per unit volume and its activity $\mathbf{Pv}$,

$$-\,\mathrm{lor}\,\mathfrak{S} = |\,P_1, P_2, P_3, \mathbf{Pv}\,|.$$

Later on its rôle was generalized for a stress, momentum and energy density of any origin, not necessarily electromagnetic,

*Cf. *Theory of Relativity*, Macmillan, 1914, Chap. IX, especially p. 238. In reproducing it here, with p_i written for g_i, I drop the imaginary unit and put $c = 1$.

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provided only that the force and its activity could be represented in the form

$$\mathbf{P} = -\nabla f - \frac{\partial \mathbf{p}}{\partial t}, \ \mathbf{Pv} = -\frac{\partial u}{\partial t} - \mathrm{div}\,\mathbf{p}.$$

From the special relativistic standpoint this array of apparently heterogeneous physical magnitudes was important as it transformed from one inertial system S to another S' as a whole, to wit by the operator $\bar{A}(\ )A$, where A is the fundamental Lorentz transformation matrix of 4×4 elements and $\bar{A}$ the transposed of A. The developed form of the transformation equations of stress-momentum-energy need not detain us here.*

The important thing in our present connection is that the said stress-momentum-energy array is a symmetrical tensor of rank two. And since such also is the contracted curvature tensor $G_{\iota\kappa}$, the idea naturally suggests itself to make $G_{\iota\kappa}$ proportional to a symmetrical covariant tensor $T_{\iota\kappa}$, of which the first nine components $T_{11}, T_{12}, \ldots T_{33}$ are of the nature of stress or equivalent to it, the components T_{4i} $(i=1, 2, 3)$ replace the momentum, and the last component T_{44} is, or approximately reduces to, an energy- or mass-density. But then it is by no means necessary (nor is it possible) to fix beforehand the exact physical meaning of the several components of such an energy tensor or *tensor of matter* (as it is often called by Einstein). Their significance has to be fixed *a posteriori*, through physical applications of the field equations aimed at.

If $T_{\iota\kappa}$ is a covariant tensor of rank two, then, as we already know,

$$T = g^{\iota\kappa}\, T_{\iota\kappa} \tag{61}$$

is a scalar, the invariant of $T_{\iota\kappa}$.† Such being the case, $g_{\iota\kappa} T$ is again a symmetrical covariant tensor. Now, guided partly by guesses (originally at least) and partly by considerations

*It will be found on p. 236 of my book quoted above.

†Such also was Laue's 'scalar' in relation to his 'Welttensor', *i.e.* the matrix S.

of conservation of energy and of momentum,* Einstein wrote down as his general *field-equations*

$$G_{\iota\kappa} = -\frac{8\pi}{c^2}\left(T_{\iota\kappa} - \tfrac{1}{2} g_{\iota\kappa} T\right), \tag{III}$$

the factor $-8\pi/c^2$ being so chosen as to give, in a first approximation, the equation of Laplace-Poisson.

In fact, as we shall see from the more definite form to be given presently, in a first approximation,

$$T_{44} = T = \rho,$$

so that the last of (III) gives $G_{44} = -\frac{4\pi}{c^2}\rho$, as in (60). Einstein's own coefficient differs from ours by the gravitation constant which is here incorporated into ρ, the density in astronomical units.

The previous equations (III°), holding outside of matter, are a special case of these general equations, for $T_{\iota\kappa} = 0$, when also $T = 0$.

To be exact, Einstein speaks first of 'matter' as 'everything except the gravitation field' (*loc. cit.*, p. 802) and writes $G_{\iota\kappa} = 0$ outside of matter in this sense of the word. But later on (p. 808), trying to justify the exact form (III) of his general equations, he states expressly that 'the energy of the gravitation field' (if there is such a thing) has also to 'act gravitationally as every energy of any other kind', in short that gravitation energy too has mass and weight. Thus, rigorously speaking, there is 'matter' everywhere, and the equations (III°) are valid nowhere, unless there is no gravitation field, when they are superfluous. In other words, gravitation itself contributes also to the tensor $T_{\iota\kappa}$. Its contribution, however, is practically evanescent, and this circumstance makes the equations (III°) physically applicable.

But even the contribution to T_{ik} $(i, k = 1, 2, 3)$ of stresses within matter in the ordinary sense of the word (tensions or pressures) is practically negligible, and so is the contribution to T_{44} of the energy proper outside of molecules, atoms or electrons, and we may as well omit it in T_{44}, and take, for a first approximation at least, $T_{44} = \rho$, where ρ is the density

*Principles to which we may return later on.

of ordinary matter or approximately so, always in astronomical units. Thus the idea easily suggests itself to build up the tensor $T_{\iota\kappa}$ for a theoretically continuous body (a fluid, liquid or solid) out of its local density and the velocity components of its motion. For although the gravitational effects of the motion of matter are exceedingly small, yet the mere desire of writing generally covariant equations, say, of hydrodynamics,* prevents us from discarding velocities in this connection. Thus, neglecting stresses, etc., let us introduce, after Einstein, the scalar or invariant ρ as 'the density' of matter, and the four-vector of velocity $\frac{dx_\iota}{ds}$. Then $\rho\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}$ will be a tensor of rank two, a contravariant one, however. Construct therefore, by the principles explained in Chapter III, the associated tensor

$$T_{\iota\kappa} = \rho\, g_{\iota\alpha}\, g_{\kappa\beta} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds}, \tag{62}$$

which will be covariant and, manifestly, a symmetrical tensor. This is Einstein's energy-tensor or tensor of matter to be used in (III) whenever tensions, pressures, etc., are negligible. As a matter of fact, in view of the limitations of even the most accurate methods of observations now available, this particular tensor will cover, presumably for many years to come, all needs of the physicist and the astronomer with regard to gravitation.†

The value of the invariant T or the scalar belonging to this tensor, which is defined by (61), follows at once. Since $g^{\iota\kappa}\, g_{\iota\alpha}\, g_{\kappa\beta} = g^{\kappa}_{\alpha}\, g_{\kappa\beta} = \delta^{\kappa}_{\alpha}\, g_{\kappa\beta} = g_{\alpha\beta}$, and $g_{\alpha\beta}\, dx_\alpha\, dx_\beta = ds^2$, we have, by (62),

$$T = \rho, \tag{62a}$$

that is to say, the scalar of the tensor in question is the density of matter.

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*And this was undoubtedly one of the reasons by which Einstein was influenced.

†The case of hydrodynamics will be covered by subtracting from (62) the tensor $pg_{\iota\kappa}$, where p is the (invariant) hydrostatic pressure.

On the other hand, in a local rest-system, in which only $(dx_4/ds)^2$ survives and is equal to $1/g_{44}$, we have

$$T_{\iota\kappa} = \rho \frac{g_{4\iota} g_{4\kappa}}{g_{44}},$$

i.e., for instance, $T_{44} = \rho g_{44}$, and therefore, by (III), rigorously,

$$G_{44} = -\frac{4\pi}{c^2} g_{44}\, \rho.$$

In a first approximation ($g_{44} \doteq 1$) this gives (60) or the Laplace-Poisson equation, as announced before.

The general field equations (III) may, independently of (62), be given a slightly different form. Multiply both sides, innerly, by $g^{\iota\kappa}$, and write

$$G = g^{\iota\kappa} G_{\iota\kappa}. \tag{63}$$

Then, since $g^{\iota\kappa} g_{\iota\kappa} = 4$,

$$G = \frac{8\pi}{c^2} T. \tag{64}$$

Substitute this value of T into (III). Then

$$G_{\iota\kappa} - \tfrac{1}{2} g_{\iota\kappa} G = -\frac{8\pi}{c^2} T_{\iota\kappa}, \tag{IIIa}$$

the required form of the field equations.

Notice that G, as defined by (63), is *the invariant* of the curvature tensor $G_{\iota\kappa}$. This invariant or rather one-sixth of it is called *the mean curvature of the world*, at the world-point in question. In fact, in the case of a three-space G would, apart from a mere numerical factor, be the arithmetical mean of the three principal riemannian curvatures, and this would still be the case for a manifold of any number of dimensions, at least if ds^2 be a definite, positive quadratic form. This justifies the name given to G above,* and equation (64), independent of the particular form (62) of $T_{\iota\kappa}$, teaches us that

*For a certain special world to be treated later on G will be proved explicitly to be *six* times the smallest value of the (constant and isotropic) curvature of three-space which it is possible to choose as a section of that four-world. (Cf. Appendix, A.)

this mean curvature is proportional to the scalar of the tensor of matter and vanishes, therefore, outside of matter. More especially, if stresses, etc., be negligible the tensor (62) comes to its right and we have, by (62a) and (64),

$$G = \frac{8\pi}{c^2}\rho. \tag{62b}$$

The mean curvature of the world is thus proportional to the density of matter.

Notice that ρ/c^2 has the dimensions of a reciprocal area, and such also are the dimensions of G, and of all $G_{\iota\kappa}$, since the $g_{\iota\kappa}$ are dimensionless, and the $G_{\iota\kappa}$ are linear in the second derivatives of the $g_{\iota\kappa}$ with respect to the coördinates, each of which is a length. The same remarks hold good with respect to the field equations (III).

It may be interesting to notice, even at this stage, that the mean curvature in familiar matter, say, in water under normal conditions, is, comparatively speaking, not insignificant. In fact, remembering that the gravitation constant is $6{\cdot}658.10^{-8}$, in c.g.s. units, we have for water at normal density

$$G = \frac{8\pi}{9.10^{20}}\,6{\cdot}658.10^{-8} = 1{\cdot}86.10^{-27}\,\mathrm{cm.}^{-2}$$

What is technically called the world-curvature is one-sixth of this. Thus the radius of mean curvature defined by $R = \sqrt{6/G}$ will be, for water,

$$R = 5{\cdot}688.10^{13}\ \mathrm{cm.},$$

i.e., about 570 million kilometers or only $3{\cdot}8$ astronomical length units.

But it would be rash to conclude with Eddington* that a globe of water of about this radius 'and no larger, could exist'. In fact, what is known from geometry is only that the total length of every straight (closed) line in a three-space of constant and *isotropic* curvature $1/R^2$, of the properly elliptic or polar kind, is πR, so that the greatest distance possible in such a space is $\frac{1}{2}\pi R$ and the total volume of the space is $\pi^2 R^3$. But, unlike such

*A S. Eddington, *Space Time and Gravitation*, Cambridge, 1920, p. 148.

a space, the world has a non-definite fundamental differential form, and its riemannian curvature depends upon the orientation of the geodesic surface element. Thus a direct transfer of the properties of an elliptic space upon the (watery) world is certainly illegitimate. Notice, moreover, that G, and therefore R, is remarkable as a genuine *invariant of the four-world* and not of a three-space laid across it as one of an infinity of possible sections. The best plan for the present is, therefore, to see in it only such an invariant of space-time, within the world-tube of a mass of water. The few numbers were here presented only to give an idea of the order of magnitude attainable by the curvature invariant in ordinary matter, considered as continuous. To those who like to contemplate sensational results the best opportunity is perhaps afforded by the atomic nuclei. According to Rutherford the radius of the nucleus of the hydrogen atom is about one-two thousandth of that of the electron, *i.e*, $\frac{1}{2}10^{-16}$ cm., and its mass practically equal to that of the whole atom. This would give for the density of mass a value 3.10^{24} times the normal water density, and therefore a curvature radius within the nucleus $\sqrt{3}\,.\,10^{12}$ smaller, *i.e.*, $R=32$ cm. only! The moral would then be that nuclei of about this radius and no larger could exist, with the same density. Fortunately they are believed to be much smaller.

But it is time to return to Einstein's equations of the gravitational field in order to see some of their further properties.

28. Multiply the field equations (IIIa), identical with (III), by $g^{\kappa\alpha}$. Then, denoting by G_ι^α and T_ι^α the mixed tensors associated with $G_{\iota\kappa}$ and $T_{\iota\kappa}$, *i.e.*, writing

$$G_\iota^\alpha = g^{\kappa\alpha}\, G_{\iota\kappa}\,, \text{ etc.},$$

and remembering that $g^{\kappa\alpha}\, g_{\iota\kappa} = g_\iota^\alpha = \delta_\iota^\alpha$, we shall have, with $\kappa = \dfrac{8\pi}{c^2}$,

$$G_\iota^\alpha - \tfrac{1}{2}\,\delta_\iota^\alpha\, G = -\kappa\, T_\iota^\alpha.$$

If $G_{\iota\lambda}^\alpha$ be the covariant derivative of G_ι^α, and similarly for the energy tensor, we have, contracting with respect to $\lambda = \alpha$, and remembering the meaning of δ_ι^α,

$$\kappa\, T_{\iota\alpha}^\alpha = \tfrac{1}{2}\,\delta_\iota^\alpha\, \frac{\partial G}{\partial x_\alpha} - G_\iota = \tfrac{1}{2}\, \frac{\partial G}{\partial x_\iota} - G_\iota.$$

But, by (56), the right hand member vanishes identically.

Thus, as a consequence of the field equations we have the four equations

$$T^{\alpha}_{\iota\alpha} = 0 \qquad (\iota = 1, 2, 3, 4), \tag{65}$$

concerning the energy tensor or the tensor of matter. Thus also, out of the ten field equations only six are left for the determination of the potentials $g_{\iota\kappa}$, as announced before.

The matter-equations, so to call (65), as a consequence of the field equations constitute a most remarkable result.* Notice that they are entirely independent of any special form of the energy tensor, such as (62). They are manifestly general, *i.e.*, valid for *any* covariant tensor $T_{\iota\kappa}$, merely in virtue of putting it equal, or proportional, to the curvature tensor

$$G_{\iota\kappa} - \tfrac{1}{2} g_{\iota\kappa} G,$$

the left hand member of Einstein's equations.

In order to see the significance of the equations (65) remember that $T^{\alpha}_{\iota\alpha}$ is the contracted covariant derivative of the tensor

$$T^{\alpha}_{\iota} = g^{\kappa\alpha}\, T_{\iota\kappa}\,.$$

Thus, if pressures, etc., are neglected, when $T_{\iota\kappa}$ has the value (62), we have, remembering that $g^{\kappa\alpha}\, g_{\iota\alpha}\, g_{\kappa\beta} = g^{\kappa}_{\iota}\, g_{\kappa\beta} = g_{\iota\beta}$,

$$T^{\alpha}_{\iota} = \rho\, g_{\iota\beta}\, \frac{dx_\beta}{ds}\, \frac{dx_\alpha}{ds}\,.$$

More generally, if we add to (62) a tensor $p_{\iota\kappa}$ due to stresses and any agents other than molar motion of matter, we shall have

$$T^{\alpha}_{\iota} = p^{\alpha}_{\iota} + \rho\, \frac{d\xi_\iota}{ds}\, \frac{dx_\alpha}{ds}\,, \tag{66}$$

where $p^{\alpha}_{\iota} = g^{\kappa\alpha}\, p_{\iota\kappa}$ is the associated supplementary tensor of matter, and

$$d\xi_\iota = g_{\iota\beta}\, dx_\beta$$

*Of course, the left hand members of the field equations were chosen by Einstein so as to yield these four equations of matter, as can be seen by comparing his earlier paper, Berlin Academy, 1915, pp. 778, 799, with a later, improved paper, *ibidem*, p. 844.

the covariant conjugate of the contravariant vector dx_α. Of the mixed tensor (66) we have to take the covariant derivative, as defined in (42a), and to contract it with respect to α and the new index. Thus the equations (65) will be

$$T^\alpha_{\iota\alpha} \equiv \frac{\partial T^\alpha_\iota}{\partial x_\alpha} - \left\{ \begin{matrix} \iota\alpha \\ \beta \end{matrix} \right\} T^\alpha_\beta + \left\{ \begin{matrix} \alpha\beta \\ \alpha \end{matrix} \right\} T^\beta_\iota = 0 \; .$$

But, as can easily be proved,

$$\left\{ \begin{matrix} \alpha\beta \\ \alpha \end{matrix} \right\} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x_\beta} , \tag{67}$$

where g is the determinant of the metrical tensor.

Thus, the four equations (65) become, in any system of coördinates,

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x_\alpha} (\sqrt{g}\, T^\alpha_\iota) - \left\{ \begin{matrix} \iota\alpha \\ \beta \end{matrix} \right\} T^\alpha_\beta = 0, \quad (\iota = 1, 2, 3, 4) \tag{65a}$$

where the mixed tensor of matter T^α_ι is as in (66). The left hand member of (65a) is a four-vector

$$T_\iota = T^\alpha_{\iota\alpha}$$

which is called *the divergence of the mixed tensor* T^α_ι. The equations (65a) can thus be read technically:

The divergence of the mixed tensor of matter vanishes.

This, of course, does not enlighten us as to their significance. To see their physical meaning take any coördinate system for which $g = -1$, so that

$$\frac{\partial T^\alpha_\iota}{\partial x_\alpha} = \left\{ \begin{matrix} \iota\alpha \\ \beta \end{matrix} \right\} T^\alpha_\beta , \tag{65b}$$

and consider the case of a weak gravitational field, for which the $g_{\iota\kappa}$ differ but little from the galilean values, *i.e.*, in quasi-cartesian coördinates, $g_{11} = -1 + \gamma_{11}$, etc., as in (21). In the expressions (66) for the energy tensor itself the $\gamma_{\iota\kappa}$ can be neglected altogether, so that

$$d\xi_\iota = -dx_\iota \quad (i = 1, 2, 3), \quad d\xi_4 = dx_4,$$

and

$$T_\iota^\alpha = p_\iota^\alpha - \rho \frac{dx_i}{ds} \frac{dx_\alpha}{ds}, \qquad (i = 1, 2, 3)$$

$$T_4^\alpha = p_4^\alpha + \rho \frac{dx_4}{ds} \frac{dx_\alpha}{ds}.$$

In the right hand member of (65*b*) the $\gamma_{\iota\kappa}$ cannot be disregarded without anihilating that member altogether. For the Christoffel symbols vanish for constant $g_{\iota\kappa}$. But since their values are taken to be small of the first order, it is enough to retain in the right hand member only

$$T_4^4 = p_4^4 + \rho \left(\frac{dx_4}{ds}\right)^2 \doteq p_4^4 + \rho,$$

and since $\left\{ {\iota 4 \atop 4} \right\} \doteq \left[{\iota 4 \atop 4} \right] = \frac{1}{2} \frac{\partial g_{44}}{\partial x_\iota}$, the four equations become, if p_4^4 be negligible in presence of ρ,

$$\frac{\partial T_\iota^\alpha}{\partial x_\alpha} = \frac{1}{2} \rho \frac{\partial g_{44}}{\partial x_\iota}.$$

For small velocities, $ds = dx_4 = cdt$, and if v_i be the cartesian velocity components dx_i/dt,

$$T_1^1 = p_1^1 - \frac{1}{c^2} \rho v_1^2, \; T_1^2 = p_1^2 - \frac{1}{c^2} \rho v_1 v_2, \text{ etc.}; \; T_1^4 = p_1^4 - \frac{\rho v_1}{c}, \text{ etc.},$$

and

$$T_4^1 = p_4^1 + \frac{\rho v_1}{c}, \; T_4^2 = p_4^2 + \frac{\rho v_2}{c}, \text{ etc.}; \; T_4^4 = p_4^4 + \rho.$$

Thus, neglecting cp_i^4 and cp_4^i in presence of the momentum (per unit volume) $\rho\mathbf{v}$ of molar motion of matter, the first three equations will be

$$\frac{\partial}{\partial x_1}(\rho v_1^2 - c^2 p_1^1) + \frac{\partial}{\partial x_2}(\rho v_1 v_2 - c^2 p_1^2) + \ldots + \frac{\partial}{\partial t}(\rho v_1) = \rho \frac{\partial \Omega}{\partial x}, \text{ etc.},$$

where $\Omega = -\frac{1}{2}c^2 g_{44}$ plays again the part of the newtonian

potential, and the fourth equation

$$\frac{\partial}{\partial x_1}(\rho v_1) + \frac{\partial}{\partial x_2}(\rho v_2) + \frac{\partial}{\partial x_3}(\rho v_3) + \frac{\partial \rho}{\partial t} = \tfrac{1}{2}\rho \frac{\partial g_{44}}{\partial t},$$

or, in obvious three-dimensional vector notation,

$$\frac{\partial \rho}{\partial t} + div\,(\rho \mathbf{v}) = -\frac{\rho}{c^2}\frac{\partial \Omega}{\partial t},$$

and, with $\mathbf{f}_1$ written for the three-vector $c^2(p_1^1,\ p_1^2,\ p_1^3)$,

$$\frac{\partial}{\partial t}(\rho v_1) + \frac{\partial}{\partial x_1}(\rho v_1^2) + \frac{\partial}{\partial x_2}(\rho v_1 v_2) + \frac{\partial}{\partial x_3}(\rho v_1 v_3) = \mathrm{div}\,\mathbf{f}_1 + \rho\frac{\partial \Omega}{\partial x_1}, \text{ etc.}$$

The left hand member of the last written equation is equal to

$$\frac{\partial}{\partial t}(\rho v_1) + v_1\frac{\partial}{\partial x_1}(\rho v_1) + \ldots + v_3\frac{\partial}{\partial x_3}(\rho v_3) + \rho v_1\,div\,\mathbf{v}$$

$$= \frac{d}{dt}(\rho v_1) + \rho v_1 \,.\, div\,\mathbf{v},$$

or, if $\delta\tau$ be the volume and $\delta m = \rho\delta\tau$ the mass of an individual element of matter, equal to $\dfrac{d(v_1\delta m)}{dt\,.\,\delta\tau}$. Similarly we have

$$\frac{\partial \rho}{\partial t} + div\,(\rho\mathbf{v}) = \frac{d\rho}{dt} + \rho\,div\,\mathbf{v} = \frac{d(\delta m)}{dt\,.\,\delta\tau}.$$

Ultimately, therefore, the approximate equations of matter are, with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as unit vectors along the coördinate axes,

$$\frac{d}{dt}(\mathbf{v}\delta m) - \delta\tau\left[\mathbf{i}\,div\,\mathbf{f}_1 + \mathbf{j}\,div\,\mathbf{f}_2 + \mathbf{k}\,div\,\mathbf{f}_3\right] = \delta m\,.\,\nabla\Omega \qquad \text{(A)}$$

and

$$\frac{d}{dt}(\delta m) = -\frac{\partial \Omega}{\partial t}\,\frac{\delta m}{c^2}. \qquad \text{(B)}$$

The first three equations embodied in the vector formula (A) are the equations of motion of a continuous medium* under internal stresses (tensions) $f_{ik} = c^2 p_i^k$, and under the action of

*A deformable solid, liquid or fluid.

the gravitational field of which the newtonian potential is again, as in the case of the approximate equations of motion of a particle, $\Omega = -\frac{1}{2}c^2 g_{44}$. The fourth equation, (B), is, apart from the new term on the right hand, the familiar equation of continuity.

In other words, the first three equations express *the principle of momentum*,—the amount of momentum acquired by matter from the field per unit time being given by δm . grad Ω which is the newtonian force on the mass element δm. And the fourth equation expresses *the principle of energy* or, equivalently, of matter,—the amount of energy ($c^2\delta m$) acquired by a material element per unit time being equal to the mass δm of that element multiplied by the local time-variation of the potential, *i.e.*, approximately equal to the decrease of the potential energy of that element.

It is scarcely necessary to say that this gain or loss in energy or in mass of 'matter' placed in a gravitational field, according to the sign of $\partial\Omega/\partial t$, is immeasurably small. Its discussion in this place may have only a mere academic interest. If it be neglected, (B) gives at once the usual equation of continuity, and (A) assumes the perfectly familiar form

$$\rho\dot{\mathbf{v}} - \mathbf{i}\ div\,\mathbf{f}_1 - \ldots = \rho\nabla\Omega = \text{gravitational force per unit volume.}$$

On the other hand it is interesting to notice that the equation (B) becomes, for $\mathbf{v}=0$, at once integrable and gives

$$\delta m = \delta m_0\, e^{-\Omega/c^2},$$

where δm_0 is δm for $\Omega = 0$. A similar result followed from a gravitation theory proposed some time ago by Nordström (*Phys. Zeits.*, 1912, p. 1126). Its interpretation may be left to the care of the reader.

Returning once more to the rigorous equations (65*a*) we now see that the terms

$$-\left\{ {\iota\alpha \atop \beta} \right\} T^\alpha_\beta$$

represent in general the momentum and energy (or mass) acquired by 'matter' in a gravitational field.

The four equations themselves express *the principles of momentum and of energy*, as was made plain above on their appropriate form. I avoid purposely to call them principles of 'conservation' of momentum and of energy. For although Einstein succeeded in giving them the form*

**Sitzungsberichte* of Berlin Academy, vol. 42, 1916, p. 1115, where the German T and t are the above $\sqrt{-g}\,T$, $\sqrt{-g}\,t$.

$$\frac{\partial}{\partial x_\alpha}\left[\sqrt{-g}\,(T_\iota^\alpha + t_\iota^\alpha)\right] = 0$$

in which they would deserve the name of conservation, yet the t_ι^α built up of the $g^{\mu\nu}$ and their first derivatives has not the character of a general tensor, but behaves so only with respect to a certain class of coördinate systems (for which $g = -1$). In view of this it has not seemed necessary to quote here the values of the t_ι^α. Suffice it to say that since, unlike T_ι^α, they contain only the gravitational potentials (and their first derivatives), Einstein calls $\sqrt{-g}\,t_\iota^\alpha$ 'the components of energy of the gravitational field', and $\sqrt{-g}\,T_\iota^\alpha$ those of matter, and reads the last set of equations: the total momentum and the total energy of matter *and* of the field are conserved. The point under consideration is after all but a formal one, and we prefer therefore to content ourselves with the original equations (65*a*), interpreting their second terms as momentum and energy gained (or lost) without attempting to locate them as such in the gravitational field before their passage to, or rather appearance in, matter.

Historically, the position is this. In the special or restricted theory of relativity the principles of conservation of momentum and of energy were expressed by the vanishing of the 'lor' or, in Laue's nomenclature, of the Divergence of a world tensor, this 'Divergence' being a four-vector whose components were transformed by the Lorentz transformation, the four equations themselves being thus invariant with respect to this kind of transformation. The tendency to imitate these principles of conservation in the generalized theory was but a most natural one. But the proper generalisation of that special Divergence in a theory admitting any transformations of the coördinates is the Divergence defined in the general tensor calculus, *i.e.*, the contracted covariant derivative

$$T_\iota \equiv T_{\iota\alpha}^\alpha$$

of the mixed tensor of matter T_ι^κ. This is a genuine four-vector, a covariant tensor of rank one, and the original generally covariant equations (65),

$$T_{\iota\alpha}^\alpha = 0,$$

are the only appropriate expression of the principles of momentum and energy. Their expanded form is (65*a*) and this cannot in general be given the form of 'conservation laws'. Only for constant $g_{\iota\kappa}$, that is in a galilean domain, does it reduce to

$$\frac{\partial}{\partial x_\alpha} T_\iota^\alpha = 0$$

which is identical with the vanishing of what was called the Divergence of T_ι^α in the restricted relativity theory. All attempts to squeeze the broader Divergence $T_{\iota\alpha}^\alpha$ into the narrower one seem artificial and useless. For conservation as an *integral* law, cf. Einstein, Berlin *Sitzungsber.*, 1918, p. 448.

29. That the gravitational field equations (together with the equations of motion and those of the electromagnetic field) can be deduced from a single variational principle or, as it is called, a *Hamiltonian Principle*, was first shown by H. A. Lorentz (Amsterdam Academy publication for 1915-16) and by D. Hilbert (Göttinger Nachrichten, 1915, No. 3), and later on by Einstein himself.* More recently Hilbert, Weyl and others have returned to this subject in a large number of publications, in some of which the importance of the Hamiltonian principle seems to be unduly overestimated.

Since this matter is, after all, of a purely formal nature, it will be enough to give here but a very brief account of Einstein's own treatment as developed in the paper just quoted.

With dx as a short symbol for $dx_1\, dx_2\, dx_3\, dx_4$ Einstein writes the Hamiltonian principle

$$\delta \int \sqrt{-g}\,(G+M)dx=0, \tag{68}$$

where G, M are invariants. Since $\sqrt{-g}\,dx$, the volume of a world-element, is invariant, so also is the whole integrand. Let M be a function of the $g_{\mu\nu}$ and of q_ι and $\partial q_\iota / \partial x_\alpha$, where q_ι are some space-time functions describing 'matter', while G is assumed to be linear in $\partial^2 g^{\mu\nu} / \partial x_\alpha\, \partial x_\beta$ with coefficients depending only upon the $g_{\mu\nu}$. Then, by partial integration,

$$\int \sqrt{g}\, G dx = \int \sqrt{g}\, G^* dx + \int F d\sigma,$$

where $d\sigma$ is an element of the boundary of the world-domain $\int dx$ (the particular value of the integrand F being irrelevant), and G^* is a function of the $g^{\mu\nu}$ and their *first* derivatives only. Let it be required that the values of $g^{\mu\nu}$ and of their first derivatives should be fixed at the boundary. Then $\delta \int F d\sigma = 0$, and we can write, instead of (68),

*A Einstein, *Hamiltonsches Prinzip und allgemeine Relativitätstheorie*, Sitzungsberichte der Akad. der Weiss., Berlin, vol. XLII, 1916, p. 1111-1116.

$$\delta \int \sqrt{g}\,(G^* + M)dx = 0, \tag{68a}$$

where the whole integrand depends upon the $g^{\mu\nu}$ and q_ι and their first derivatives only. Thus the variation of the $g^{\mu\nu}$ gives at once the ten equations

$$\frac{\partial}{\partial x_\alpha}\frac{\partial H^*}{\partial p_{(\alpha)}} - \frac{\partial H^*}{\partial p} = \frac{\partial N}{\partial p}, \tag{69}$$

where

$$H^* = \sqrt{-g}\,G^*, \quad N = \sqrt{-g}\,M$$

and

$$p = g^{\mu\nu}, \ p_{(\alpha)} = \frac{\partial p}{\partial x_\alpha}.$$

Now, let G be the curvature invariant, *i.e.*, in our previous notation,

$$G = g^{\iota\kappa}\,G_{\iota\kappa}.$$

Then, on performing the said partial integration, it will be found that

$$H^* = \sqrt{-g}\,g^{\mu\nu}\left[\begin{Bmatrix}\mu\alpha\\ \beta\end{Bmatrix}\begin{Bmatrix}\nu\beta\\ \alpha\end{Bmatrix} - \begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix}\begin{Bmatrix}\alpha\beta\\ \beta\end{Bmatrix}\right]. \tag{70}$$

With this value of H^* *the equations* (69) *become identical with Einstein's field equations* as given above, if we put

$$g^{\mu\nu}\frac{\partial N}{\partial g^{\mu\sigma}} = -\sqrt{-g}\,T^\nu_\sigma,$$

i.e.,

$$g_{\nu\beta}\,g^{\mu\nu}\frac{\partial N}{\partial g^{\mu\sigma}} = -\sqrt{-g}\,g_{\nu\beta}\,T^\nu_\sigma$$

or, in terms of the covariant tensor of matter,

$$\frac{\partial M}{\partial g^{\beta\sigma}} = -T_{\beta\sigma}. \tag{71}$$

The verification of this statement may be left to the care of the reader who may confine himself to systems for which $g = -1$.

30. *Gravitational waves.* Let us close this chapter by briefly mentioning a method of approximate integration of the field-equations given by Einstein (Berlin Academy proceedings for 1916, p. 688) which exhibits the propagation of gravitational disturbances.

Let the $g_{\iota\kappa}$ differ but little from the galilean values, in a cartesian system, say, or—in our previous notation—let

$$g_{\iota\kappa} = \bar{g}_{\iota\kappa} + \gamma_{\iota\kappa} \,,$$

where the $\gamma_{\iota\kappa}$ are small. Then Einstein's approximate solution of his field-equations is

$$\gamma_{\iota\kappa} = \gamma'_{\iota\kappa} - \tfrac{1}{2}\, \delta^{\kappa}_{\iota} \gamma'_{\lambda\lambda} \,, \tag{72a}$$

where $\gamma'_{\iota\kappa}$ is *the retarded potential* of $-2\kappa\, T_{\iota\kappa}$, that is to say, the familiar particular solution

$$\gamma'_{\iota\kappa} = -\frac{2\kappa}{4\pi} \iiint \frac{1}{r}\, T_{\iota\kappa}\, (x, y, z, ct-r) dx\, dy\, dz \tag{72b}$$

of 'the wave-equation'

$$\left(\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \nabla^2\right) \gamma'_{\iota\kappa} = -2\kappa\, T_{\iota\kappa} \,. \tag{72c}$$

In (72*b*) r is the three-dimensional distance of the point for which $\gamma'_{\iota\kappa}$ is required (for the instant t) from the integration element $dxdydz$ at which the value of $T_{\iota\kappa}$ prevailing at the instant $t - \dfrac{r}{c}$ is to be taken.

This solution represents gravitation as being propagated with the normal light velocity c, the slight changes of the latter due to the gravitational field itself being manifestly neglected. In this approximation the rigorously *non-linear* field equations are replaced by linear differential equations of the form (72*c*), the usual wave-equations.

In the sub-case of a stationary gravitational field, when the whole tensor of matter is reduced to $T_{44} = \rho$, we have by (72*b*), as the only surviving $\gamma'_{\iota\kappa}$,

$$\gamma'_{44} = -\frac{\kappa}{2\pi} \iiint \frac{\rho dx\, dy\, dz}{r} = -\frac{\kappa\Omega}{2\pi} \,,$$

where Ω is the ordinary newtonian potential of the gravitating masses, and, by (72a), the only surviving $\gamma_{\iota\kappa}$,

$$\gamma_{11} = \gamma_{22} = \gamma_{33} = -\gamma_{44} = \frac{\kappa\Omega}{4\pi} = \frac{2\Omega}{c^2},$$

so that, as before, the role of the potential Ω is taken over in part by $-\frac{1}{2}c^2\gamma_{44}$.

CHAPTER V.

Radially Symmetric Field. Perihelion Motion, Bending of Rays, and Spectrum Shift.

31. In order to represent the motion around the sun of a planet as a 'free particle', of mass negligible compared to that of the central body, it is enough to find a *radially symmetrical* solution of Einstein's field equations outside the sun,

$$G_{\iota\kappa} = 0, \tag{III°}$$

considering the origin $r=0$ of polar coördinates r, ϕ, θ as a singular point.

As a form of the line-element, sufficiently general for this purpose, let us assume

$$ds^2 = g_1\, dr^2 - r^2 d\phi^2 - r^2 \sin^2\phi\, d\theta^2 + g_4\, c^2 dt^2, \tag{73}$$

where g_1, g_4, written instead of g_{11}, g_{44}, are functions of r *alone*, of which we shall thus far assume only that

$$g_1(\infty) = -1, \quad g_4(\infty) = 1, \tag{74}$$

i.e., that at distances r large compared with a certain length belonging to the sun (which will appear in the sequel) the line-element tends to its galilean form $ds^2 = -dr^2 - r^2(d\phi^2 + \sin^2\theta d\phi^2) + c^2dt^2$.

Let us correlate the indices of the coördinates by putting

$$x_1, x_2, x_3, x_4 = r, \phi, \theta, ct$$

respectively. Then the metrical tensor in question will consist of the components

$$g_1 = g_1(r),\ g_2 = -r^2,\ g_3 = -r^2 \sin^2\phi,\ g_4 = g_4(r), \tag{73a}$$

where g_κ has been written for $g_{\kappa\kappa}$.

In the more general case

$$ds^2 = g_\kappa\, dx_\kappa^{\ 2},$$

in which the $g_\kappa = g_{\kappa\kappa}$ are *any* functions of all the variables, we have, for the only surviving associated tensor components,

$$g^{\kappa\kappa} = \frac{1}{g_\kappa},$$

and therefore, recalling the definition of the Christoffel symbols,

$$\left.\begin{aligned}
\left\{\begin{matrix} \iota\kappa \\ \kappa \end{matrix}\right\} &= \frac{1}{2g_\kappa}\frac{\partial g_\kappa}{\partial x_\iota}, \text{ for all } \iota, \kappa, \\
\left\{\begin{matrix} \iota\iota \\ \kappa \end{matrix}\right\} &= -\frac{1}{2g_\kappa}\frac{\partial g_\iota}{\partial x_\kappa}, \text{ for } \iota \pm \kappa,
\end{aligned}\right\} \tag{75}$$

while all other Christoffel symbols vanish.

Applying these formulae to the more special tensor (73*a*), writing

$$h_1 = \log g_1, \; h_4 = \log g_4$$

and using dashes for derivatives with respect to $r = x_1$, we have the rigorous values of the only surviving Christoffel symbols, altogether nine in number,

$$\left\{\begin{matrix} 11 \\ 1 \end{matrix}\right\} = \tfrac{1}{2} h_1', \left\{\begin{matrix} 12 \\ 2 \end{matrix}\right\} = \left\{\begin{matrix} 13 \\ 3 \end{matrix}\right\} = \frac{1}{r}, \left\{\begin{matrix} 14 \\ 4 \end{matrix}\right\} = \tfrac{1}{2} h_4'$$

$$\left\{\begin{matrix} 22 \\ 1 \end{matrix}\right\} = \frac{r}{g_1}, \left\{\begin{matrix} 23 \\ 3 \end{matrix}\right\} = \cot\phi$$

$$\left\{\begin{matrix} 33 \\ 1 \end{matrix}\right\} = \frac{r}{g_1}\sin^2\phi, \left\{\begin{matrix} 33 \\ 2 \end{matrix}\right\} = -\tfrac{1}{2}\sin 2\phi; \; \left\{\begin{matrix} 44 \\ 1 \end{matrix}\right\} = -\frac{1}{2g_1} g_4'.$$

These values substituted into the general expressions (55) for the components $G_{\iota\kappa}$ of the curvature tensor give *zeros* for all those having $\iota \neq \kappa$, while the remaining four diagonal components are, rigorously,

$$\left.\begin{aligned}
G_{11} &= \tfrac{1}{2} h_4'' + \tfrac{1}{4} h_4'(h_4' - h_1') - \frac{h_1'}{r} \\
G_{22} &= -1 - \frac{1}{g_1}\left[1 + \frac{r}{2}((h_4' - h_1')\right] \\
G_{33} &= \sin^2\phi \,.\, G_{22} \\
G_{44} &= \frac{g_4}{g_1}\left[G_{11} + \frac{1}{r}(h_1' + h_4')\right].
\end{aligned}\right\} \tag{77}$$

Thus we have, according to the field equations for $r>0$, that is, outside of matter, for the two unknown functions g_1, g_2 the three differential equations

$$(a) \quad h_4''+\tfrac{1}{2}h_4'(h_4'-h_1')-\frac{2h_1'}{r}=0$$

$$(b) \quad \frac{r}{2}(h_4'-h_1')+g_1+1=0$$

$$(c) \quad h_1'+h_4'=0.$$

The last of these equations gives $h_1+h_4=\log(g_1 g_4)=\text{const.}$, that is to say, by (74),

$$g_1 g_4 = const. = -1.$$

Equation (b) now becomes $g_4+rg_4'=1$ or

$$\frac{d}{dr}\log[r(g_4-1)]=0,$$

so that $r(g_4-1)=\text{const.}=-2L$, say.

Thus the rigorous, and the most general, solution of the field equations (b), (c) is ultimately,

$$g_4=1-\frac{2L}{r}, \quad g_1=-\left(1-\frac{2L}{r}\right)^{-1}, \tag{78}$$

where L is any constant, some *length,* characterising the sun, *i.e.*, here the singular point or centre of the gravitation field. As to the first field equation (a) it is satisfied identically by these values of g_1, g_4.*

To express the constant L in terms of M, the sun's mass in astronomical units, we may apply the following reasoning: As we already know, in the approximate equations the newtonian potential $\Omega=M/r$ is replaced by $-\frac{c^2}{2}\gamma_{44}$. Now,

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*In fact, since $g_1g_4=-1$, the left hand member of equation (a) becomes

$$h_4''+h_4'^2+2h_4'/r,$$

and this is, by (78),

$$\frac{L}{r^2(r-2L)^2}\,[4L-2r+2(r-2L)],$$

which vanishes identically for all $r\neq 2L$.

in the present case, $\gamma_{44}=g_4-1=-2L/r$. Thus $M=c^2L$, whence

$$L=\frac{M}{c^2}. \tag{79}$$

Ultimately therefore, the line-element (73) corresponding to a radially symmetrical field becomes, rigorously,

$$ds^2=\left(1-\frac{2L}{r}\right)c^2dt^2-\left(1-\frac{2L}{r}\right)^{-1}dr^2-r^2(d\phi^2+\sin^2\phi d\theta^2), \tag{80}$$

a form of the solution of the field equations first given by Schwarzschild (Berlin Academy proceedings, 1916, p. 189). As was already mentioned in Chapter IV, the dimensions of L as defined by (79) are those of a length. This length, which is sometimes called *the gravitation radius* of the given body, amounts for our sun to about 1·47 kilometers. Thus, for all applications of any actual interest, L/r is a small fraction and the coefficient of dr^2 can be replaced by $-(1+2L/r)$.

32. *Perihelion motion.* Let us now consider the motion of a free particle (planet) in the field determined by the line-element (80), that is to say, by the metrical tensor

$$g_1=-\left(1-\frac{2L}{r}\right)^{-1},\ g_2=-r^2,\ g_3=-r^2\sin^2\phi,\ g_4=-\frac{1}{g_1}.$$

The general equations of motion (15) with the values (76) of the Christoffel symbols become, for $\iota=2, 3, 4$,*

$$\frac{d^2\phi}{ds^2}=-\frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds}+\tfrac{1}{2}\sin 2\phi\left(\frac{d\theta}{ds}\right)^2$$

$$\frac{d^2\theta}{ds^2}=-\left[\frac{2}{r}\frac{dr}{ds}+\cot\phi\frac{d\phi}{ds}\right]\frac{d\theta}{ds}$$

$$\frac{d^2x_4}{ds^2}=-h_4'\frac{dr}{ds}\frac{dx_4}{ds}.$$

*Instead of the first equation of motion ($\iota=1$) it will be more convenient to take the identical equation $g_{\iota\kappa}\dot{x}_\iota\dot{x}_\kappa=1$.

Lay the plane $\phi=\pi/2$, the equatorial plane of the coördinate system, through the direction of motion of the planet at some instant t_0. Then, at that instant, $d\phi/ds=0$ and $\sin 2\phi=0$, and therefore, by the first equation, permanently $\phi=\pi/2$, that is to say, the planet will describe a *plane* orbit, and the remaining two equations, together with the identical equation $g_{\iota\kappa}\dot{x}_\iota\dot{x}_\kappa=1$, will become

$$\ddot{\theta}+\frac{2\dot{r}}{r}\dot{\theta}=0,$$

$$\ddot{x}_4+h_4'\dot{r}\dot{x}_4=0,$$

$$g_4\dot{x}_4^2-\frac{1}{g_4}\dot{r}^2-r^2\dot{\theta}^2=1, \tag{80a}$$

where $h_4=\log g_4$ and $g_4=1-2L/r$. The first two of these equations can be written

$$\frac{d}{ds}\log(r^2\dot{\theta})=0,\quad \frac{d}{ds}\log(g_4\dot{x}_4)=0,$$

and give

$$r^2\dot{\theta}=p,\ g_4\dot{x}_4=k, \tag{81}$$

where p and k are arbitrary constants.* With the values of $\dot{x}_4$ and $\dot{\theta}$ derived from (81) equation (80a) becomes

$$\frac{p^2}{r^4}\left(\frac{dr}{d\theta}\right)^2+\left(1+\frac{p^2}{r^2}\right)\left(1-\frac{2L}{r}\right)=k^2$$

or, putting $\rho=\dfrac{1}{r}$,

$$\left(\frac{d\rho}{d\theta}\right)^2=\frac{k^2-1}{p^2}+\frac{2L}{p^2}\rho-\rho^2+\underline{2L\rho^3}. \tag{82}$$

The determination of the orbit is thus reduced to a quadrature. As an alternative we may write down the differential equation of the orbit, by differentiating the last equation with respect to θ,

*The first of (81) represents the slightly modified law of Kepler: areas swept out by the radius vector in equal *proper times* of the particle (s/c instead of t) are equal.

$$\frac{d^2\rho}{d\theta^2}+\rho=\frac{L}{p^2}+\underline{3L\rho^2.} \tag{82a}$$

Either equation differs from the familiar equations of celestial mechanics, based on Newton's principles, only by the underlined last term of the right-hand member.

It is well known that in the absence of this supplementary term the orbit is a conic (an ellipse, a parabola or a hyperbola)

$$\rho=\frac{L}{p^2}\left[1+\epsilon\cos(\theta-\tilde{\omega})\right] \tag{83}$$

with fixed perihelion, $\tilde{\omega}=$const. In fact, equation (82*a*) is identically satisfied by (83); and so is (82) if we put

$$p^2(1-\epsilon^2)=L^2(1-k^2), \tag{83a}$$

so that the orbit is an ellipse, a parabola or a hyperbola according as k^2 is smaller, equal to or greater than 1.

In general, for orbital velocities comparable with the light velocity, equation (82) gives θ as an elliptic integral of ρ, to which corresponds a complicated non-closed orbit. Its discussion may be left to the care of the reader.* Here it will be enough to consider small velocities such as occur among the planets of the solar system. The supplementary term is then small compared with the newtonian ones, and the problem can be solved approximately by a conic (83) with slowly moving perihelion. If $\partial\rho/\partial\theta$ is the derivative of ρ when $\tilde{\omega}$ is kept fixed, and if the term with $(d\tilde{\omega}/d\theta)^2$ is neglected, we have

$$\left(\frac{d\rho}{d\theta}\right)^2=\left(\frac{\partial\rho}{\partial\theta}+\frac{\partial\rho}{\partial\tilde{\omega}}\,\frac{d\tilde{\omega}}{d\theta}\right)^2\doteq\left(\frac{\partial\rho}{\partial\theta}\right)^2+2\frac{\partial\rho}{\partial\theta}\,\frac{\partial\rho}{\partial\tilde{\omega}}\,\frac{d\tilde{\omega}}{d\theta},$$

and since $(\partial\rho/\partial\theta)^2$ itself accounts for the first three terms of the right hand member of (82), the perihelion motion will be determined by†

*Cf. A. R. Forsyth, *Proc. Roy. Soc.*, XCVII (1920), p. 145, also W. B. Morton, *Phil. Mag.*, XLII (1921), p. 511.

†This reasoning, aiming at the *secular* motion of the perihelion, presupposes the knowledge of absence of a secular variation of the eccentricity ϵ. Cf. footnote on p. 99, *infra*.

$$\frac{\partial \rho}{\partial \theta} \frac{\partial \rho}{\partial \tilde{\omega}} \frac{d\tilde{\omega}}{d\theta} = L\rho^3.$$

Here (83) can be used with sufficient accuracy for ρ and its two derivatives, so that

$$\frac{d\tilde{\omega}}{d\theta} = -\frac{L^2}{p^2\epsilon^2} \cdot \frac{1+3\epsilon\cos u+3\epsilon^2\cos^2 u+\epsilon^3\cos^3 u}{\sin^2 u},$$

where $u=\theta-\tilde{\omega}$. Integrating this from 0 to 2π over θ or, what for our approximation is the same thing, over u, we shall have the secular perturbation $\delta\tilde{\omega}$, the motion of the perihelion *per period of revolution*. The second and the third terms of the integrand, having in the second and the third quadrants values opposite to those in the first and fourth, contribute nothing to the secular perihelion motion, and the same is true of the first term, since this is the derivative of the periodic function $-\cot u$. We are thus left with

$$\delta\tilde{\omega} = -\frac{3L^2}{p^2}\int_0^{2\pi} \cot^2 u\, du,$$

and since $-\cot^2 u$ is the derivative of $\cot u+u$,

$$\delta\tilde{\omega} = \frac{6\pi L^2}{p^2}. \tag{84}$$

This being essentially positive, the secular motion of the perihelion is *progressive*, that is, in the sense of the revolution of the planet.

If the orbit be an ellipse ($\epsilon^2<1$) with semi-axes a, b, we have, by the original meaning of the constant p,

$$p = r^2 \frac{d\theta}{ds} \doteq \frac{r^2}{c}\frac{d\theta}{dt} = \frac{2\pi ab}{cT},$$

where T is the period of revolution, and by (83),

$$L = \frac{M}{c^2} = p^2\frac{a}{b^2} = \frac{4\pi^2 a^3}{c^2T^2},$$

expressing Kepler's third law. Thus

$$\frac{\lambda}{p} = \frac{2\pi a^2}{cTb} = \frac{2\pi a}{cT\sqrt{1-\epsilon^2}},$$

and (84) becomes

$$\delta\tilde{\omega} = \frac{24\pi^3 a^2}{c^2T^2(1-\epsilon^2)}, \tag{84a}$$

which is Einstein's formula for the secular motion of the perihelion of a planet, undisturbed by other planets, per period of revolution.* This formula gives for Mercury, per century, 43″ or 43″·1, coinciding most remarkably with the famous excess of perihelion motion of that planet, unaccounted for by the perturbations due to the other members of the solar family of celestial bodies. Although the rival explanation based on perturbing zodiacal matter, due to Seeliger—Newcomb (taken up more recently by Harold Jeffreys), cannot be considered as ultimately discarded, this is certainly a most conspicuous achievement, perhaps the greatest triumph of Einstein's theory, yielding the required excess without the aid of any new empirical constant in addition to the light velocity and the gravitation constant. As to the remaining planets, Einstein's formula gives for them secular perihelion motions too small to be either contradicted or confirmed by observation in the present state of the astronomer's knowledge. In fact, the only other serious anomaly unaccounted for by newtonian celestial mechanics (unless Seeliger's theory is accepted) is the excessive motion of the nodes of Venus, but with this Einstein's theory is essentially powerless to deal, since it yields, for a radially symmetric centre of course, rigorously plane orbits. But even the outstanding node motion of Venus is generally felt to be much less important than Mercury's perihelion motion yielded so naturally by Einstein's theory of gravitation.

*A more thorough analysis shows that this is *the only* secular perturbation, the eccentricity, the period and the remaining elements of planetary motion being unaffected by the deviation of Einstein's theory from that of classical celestial mechanics. Cf. W. de Sitter's paper in *Monthly Notices of the Roy. Astr. Soc.*, London, 1916, pp. 699 *et seq.*, more especially section 17.

33. *Deflection of light rays.* The propagation of light is given by the minimal lines $ds=0$ of the metrical manifold determined by the quadratic form (80). By reasons of symmetry it is again sufficient to consider the plane $\phi=\text{const.}=\pi/2$. Thus the light equation becomes

$$\frac{1}{g_4}\,dr^2+r^2d\theta^2=g_4c^2\,dt^2,\quad g_4=1-\frac{2L}{r}.$$

If v be the system-velocity or the 'coördinate velocity' of light, defined by

$$v^2=\left(\frac{d\sigma}{dt}\right)^2=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2,$$

the preceding equation gives

$$\left[\frac{1}{g_4}\left(\frac{dr}{d\sigma}\right)^2+r^2\left(\frac{d\theta}{d\sigma}\right)^2\right]v^2=c^2g_4,$$

and if η be the angle between the tangent to the light ray and the radius vector, so that $dr/d\sigma=\cos\eta$, $rd\theta/d\sigma=\sin\eta$,

$$\frac{c^2}{v^2}=\frac{1}{g_4}\left[\frac{\cos^2\eta}{g_4}+\sin^2\eta\right]. \tag{85}$$

Thus, if the ray be radial, away from or towards the origin, the light velocity is cg_4, and if transversal, $c\sqrt{g_4}$, both principal velocities being smaller than c, and both tending to c at infinity. Neglecting the square and the higher powers of L/r, which even at the surface of the sun is a very small fraction, we can write, approximately, $v/c \doteq 1-(1+\cos^2\eta)L/r$.

The velocity of light being determined by (85), the shape of the ray or the light path between any two points 1, 2 can be found* by means of Fermat's principle

$$\delta\int_1^2 dt=\delta\int_1^2\frac{d\sigma}{v}=0.$$

In fact this principle can be proved to hold, at least for stationary gravitational fields, *i.e.*, for $g_{\iota\kappa}$ not containing the

*In terms of r, η, and thence by integration in terms of r, θ.

time* as in the case in hand. Those interested in such an application of Fermat's principle may consult de Sitter's paper quoted in the preceding section.

But a much more speedy way of obtaining the light path is to consider it as the limiting case of the orbit of a free particle. In fact, returning to the differential equation (82a) of such an orbit, and remembering the original meaning of the integration constant p,

$$p=r^2\frac{d\theta}{ds},$$

we have for light, or for a 'particle' which would everywhere keep pace with it,

$$p=\infty,$$

so that the differential equation of the light path becomes

$$\frac{d^2\rho}{d\theta^2}+\rho-3L\rho^2=0. \tag{86}$$

In the absence of the last term, which bears to ρ the small ratio $3L/r$, we should have $\rho=\rho_0\cos\theta$, a straight line whose shortest distance from the origin is $r_0=1/\rho_0$, the angle θ being measured from the corresponding radius vector. Thus, replacing ρ in the last term by $\rho_0\cos\theta$, which amounts to neglecting $L^2\rho^2$-and higher order terms, we have for the light ray $\rho=\rho_0\left[\cos\theta+L\rho_0(1+\sin^2\theta)\right]$ or

$$\frac{r_0}{r}=\cos\theta+\frac{L}{r_0}(1+\sin^2\theta).$$

The angle between the asymptotes ($r/r_0=\infty$) of this curve is easily found to be

$$\Delta=\frac{4L}{r_0}=\frac{4M}{c^2r_0}. \tag{87}$$

This is then the total *angle of deflection* of a light ray arriving

*For a simple proof see T. Levi-Civita's paper in *Nuovo Cimento*, vol. XVI, 1918, p. 105. Levi-Civita assumes also $g_{14}=g_{24}=g_{34}=0$. The latter limitation, however, does not seem to be necessary. Thus, for instance, it can be shown that Fermat's principle leads to a correct result in the case of a uniformly rotating system, *i.e.*, obtained from a galilean system by the transformation $\theta'=\theta+\bar{\omega}t$, $\bar{\omega}=\text{const.}$

from a distant source (star) to the earth, if r_0 be, approximately, the shortest distance of the ray from the origin, *e.g.*, from the sun's centre. In the latter case, if R be the sun's radius, we have $4L/R = 5{\cdot}88/6{\cdot}97.10^5$ radians $= 1''{\cdot}75$, so that

$$\Delta = 1''{\cdot}75 \frac{R}{r_0} .$$

This is Einstein's famous formula for the displacement of star images seen in comparative angular proximity to the sun's disc. It can be considered as fairly well verified by the results of the Eclipse Expedition at Sobral, Brazil,* of May 29, 1919, which were ultimately estimated to give, when reduced to $r_0 = R$, the value $1''{\cdot}98$ with a probable error of about six per cent. This is certainly more than a mere order-of-magnitude coincidence, and speaks strongly in favour of Einstein's theory.

The displacements according to Einstein's formula should, of course, be away from the sun and purely *radial.* The displacements measured on the Sobral plates deviated from radial directions, at least for four out of the seven stars, considerably, to wit by 35°, 16°, 8°, and 6° for the stars numbered 11, 6, 2, and 10, whose distances from the sun's centre were about 8, 4, 2, and $5R$ respectively. These deviations or the presence of transversal displacement components may well be due to the distortion of the coelostat-mirror by the sun's heat, as pointed out by Prof. H. N. Russell. Yet a refined investigation of this point during the next eclipse seems very desirable, and, as I understand, will be taken special care of at the Eclipse Expedition of September 20, 1922, at which it is designed to avoid the use of a mirror. The field of stars near the sun, during totality, will then be almost as favourable as in 1919.†

34. *Shift of spectrum lines.* Consider an atom, say of nitrogen, placed in the photosphere of the sun, at rest or practically so. Then its line-element or the element of its 'proper time' will be, by (80), and writing for the present s instead of s/c,

$$ds = dt\,\sqrt{g_{44}} = dt\left(1 - \frac{2L}{R}\right)^{\frac{1}{2}},$$

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*The measurements of the Principe Expedition, made under unfavourable weather conditions, seem by far less reliable.

†Some preliminary details will be found in *Monthly Notices* of the Roy. Astr. Soc. for May 1920, p. 628.

and any finite interval of its proper time

$$\Delta s=\left(1-\frac{2L}{R}\right)^{\frac{1}{2}}\Delta t \doteq \left(1-\frac{L}{R}\right)\Delta t.$$

Let another nitrogen atom be placed in one of our terrestrial laboratories, at a distance r from the sun's centre. Then its proper-time interval will be

$$\Delta s_1=\left(1-\frac{L}{r}\right)\Delta t_1.$$

In particular, let Δt_1 be the terrestrial, and Δt the solar time period of one of the natural vibrations or spectrum lines of nitrogen.

Now, encouraged by the traditional belief in the somewhat vague 'sameness' of atoms of a given kind, Einstein assumes, as he did already in other circumstances in the special relativity theory, that the said two atoms are 'equal' to each other in the sense of the word that *the proper times** of their vibration periods are equal to each other. Eddington in his *Report* (p. 56) simply says that an atom *is* "a natural clock which ought to give an invariant measure of an interval δs, *i.e.*, the interval δs corresponding to one vibration of the atom is always the same". Weyl states the case in an apparently more profound way by saying that if the two atoms are "*objectively equal* to each other, the process by which they emit waves of a spectrum line, when measured by the *proper time,* must have in both the same frequency".

In short, the founder of the theory, as well as his exponents assume, more or less implicitly, that

$$\Delta s=\Delta s_1.$$

If so, then the ratio of the solar to the terrestrial period of vibrations is

$$\frac{\Delta t}{\Delta t_1}=\left(1-\frac{L}{r}\right):\left(1-\frac{L}{R}\right),$$

or, since in our case R/r is but a small fraction,

$$\frac{\Delta t}{\Delta t_1}=1+\frac{L}{R}=1+2{\cdot}109.10^{-6}. \qquad (88)$$

*It is now usual to extend this name for ds/c from special to general relativity theory.

Einstein's conclusion then is that the lines of the solar spectrum, compared with those of a terrestrial one, should be *shifted towards the red*, the proportionate increment of wave-length being

$$\frac{\delta\lambda}{\lambda} = \frac{L}{R} = 2{\cdot}109.10^{-6},$$

or equivalent to a Doppler effect due to a (receding) source velocity of 0·633 kilometers per second. This amounts, for violet light, to about 0·008 Å. Now, although with the modern means one-thousandth of an Å or even less can be well detected in comparing spectra, Dr. St. John of the Mount Wilson Observatory, who observed 43 lines of nitrogen (cyanogen) at the sun's centre, and 35 at the limb, was unable to detect any trace of the predicted effect. His observations were made and discussed in 1917, and his final conclusion then was that "there is no evidence of a displacement, either at the centre or at the limb of the sun, of the order 0·008 Å". Since that time, however, in view of the entanglement of the Einstein effect with shifts of a different origin, and seeing that the results of other astrophysicists were not quite so definite, Dr. St. John suspended his final judgment and is now taking up a thorough discussion of the whole material of solar spectrum shifts from E. L. Jewell's first observations, made about 1890, up to the present. The natural impression now is that it would be premature to either assert or deny the existence of the gravitational spectrum shift.

Einstein himself has, on more than one occasion, expressed the very radical opinion that, should the shift be absent, the whole theory should be abandoned. Yet, in view of the hypothetical nature of *the sameness* of atoms in the explained sense of the word, such an attitude, though personally intelligible, is by no means necessary. It is true that the invariability of an atomic *s*-period of vibration in a gravitational field can, with the aid of the equivalence hypothesis, be reduced to its invariability while the atom is being moved about,—a property of atoms as 'natural clocks' already

utilised in special relativity.* Yet we do not know whether the atoms actually possess even the latter property. Thus, Einstein's intransigent attitude proves only the strength of his belief that the atoms are or will turn out to be such natural, ideal clocks. But, after all, this is only a guess. A very reasonable one to be sure; for if not among the atoms, then there is indeed but little hope to find such clocks among other 'mechanisms', natural or artificial.

At any rate, a final astrophysical verification of Einstein's spectrum-shift formula, supported perhaps by repeated experiments on canal rays, would be an achievement of fundamental importance. Until then 'the natural clock' will remain a purely abstract concept.

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*It is this theoretical attribute of atoms which has led to the conclusion that moving hydrogen atoms (canal rays) will emit, in transversal directions, waves $(1-v^2/c^2)^{-\frac{1}{2}}$ times longer than atoms at rest. But even this shift effect, though tried experimentally, does not seem to have ever been detected.

CHAPTER VI

Electromagnetic Equations

35. Maxwell's equations of the electromagnetic field in empty space supplemented by the convection current $\rho\mathbf{v}$, or the fundamental equations of the electron theory are, in three-dimensional vector notation, with $x_4 = ct$,

$$\frac{\partial \mathbf{M}}{\partial x_4} + \operatorname{curl} \mathbf{E} = 0, \quad \operatorname{div} \mathbf{M} = 0$$

$$-\frac{\partial \mathbf{E}}{\partial x_4} + \operatorname{curl} \mathbf{M} = \rho \frac{\mathbf{v}}{c}, \quad \operatorname{div} \mathbf{E} = \rho.$$

They contain, apart from the velocity $\mathbf{v}$ of moving charges, but two vectors $\mathbf{E}$, $\mathbf{M}$ which may be provisionally called the electric and the magnetic forces. As is well-known from the special relativity theory, these equations retain their form or are covariant with respect to the Lorentz transformation, *i.e.*, in passing from one to another inertial system.*

They are not, however, generally covariant, and thus not appropriate to the purposes of the general relativity theory.

What is covariant with respect to any coördinate transformations is the somewhat broader system of equations, containing two more vectors $\mathbf{D}$ and $\mathbf{B}$ which may be called the electric and the magnetic polarizations,†

$$\frac{\partial \mathbf{B}}{\partial x_4} + \operatorname{curl} \mathbf{E} = 0, \quad \operatorname{div} \mathbf{B} = 0, \tag{A}$$

$$-\frac{\partial \mathbf{D}}{\partial x_4} + \operatorname{curl} \mathbf{M} = \rho \frac{\mathbf{v}}{c}, \quad \operatorname{div} \mathbf{D} = \rho. \tag{B}$$

*Cf. for instance my *Theory of Relativity*, 1914, Chap. VIII, and, for the historical aspect of the subject, Chap. III.

†Or the electric displacement and the magnetic induction respectively.

In a galilean domain or an inertial system **D** and **B** reduce to **E** and **M** respectively, but in general, in a gravitational field or a non-inertial system, the polarizations differ from the forces, being some linear vector functions of the latter.

The general covariance of these two groups of electromagnetic equations was first noticed and developed by F. Kottler as early as in 1912* and shortly afterwards, with due acknowledgement, incorporated by Einstein into the physical part of his general theory of relativity.

Let $F_{\iota\kappa}$ be an antisymmetric covariant tensor of rank two or a six-vector, which will embody in itself **B** and **E**, and thus may be called *the magneto-electric six-vector.* Then the group (A) of equations can be replaced by the equations

$$\frac{\partial F_{\iota\kappa}}{\partial x_\lambda}+\frac{\partial F_{\kappa\lambda}}{\partial x_\iota}+\frac{\partial F_{\lambda\iota}}{\partial x_\kappa}=0, \qquad (A_1)$$

which are generally covariant since their left hand members are, by (46), Chap. III, the components of a general tensor of rank three, the antisymmetric *expansion* of the six-vector $F_{\iota\kappa}$.

To compare (A_1) with (A) and to see the simplest form of the correlation between **B**, **E** and the six components of $F_{\iota\kappa}$ use cartesian coördinates or, in the presence of a gravitational field (always 'weak'), quasi-cartesian coördinates and denote by 1, 2, 3 the rectangular components of **B**, **E** along the three axes. Then the group (A) of equations will be

$$\frac{\partial B_1}{\partial x_4}+\frac{\partial E_3}{\partial x_2}-\frac{\partial E_2}{\partial x_3}=0, \text{ etc.}$$

$$\frac{\partial B_1}{\partial x_1}+\frac{\partial B_2}{\partial x_2}+\frac{\partial B_3}{\partial x_3}=0,$$

where 'etc.' means two more equations by cyclic permutation of the suffixes 1, 2, 3 only. On the other hand, writing out (A_1) and remembering that $F_{\iota\kappa}=-F_{\kappa\iota}$, we have

*Friedrich Kottler, *Raumzeitlinien der Minkowski' schen Welt*, Sitzungsberichte Akad. Wien, vol. 121, section II*a*, pp. 1659-1759.

$$\frac{\partial F_{23}}{\partial x_4}+\frac{\partial F_{34}}{\partial x_2}-\frac{\partial F_{24}}{\partial x_3}=0, \text{ etc.}$$

$$\frac{\partial F_{23}}{\partial x_1}+\frac{\partial F_{31}}{\partial x_2}+\frac{\partial F_{12}}{\partial x_3}=0,$$

and these four equations become identical with those just written if we put

$$F_{23},\ F_{31},\ F_{12}=B_1,\ B_2,\ B_3$$
$$F_{14},\ F_{24},\ F_{34}=E_1,\ E_2,\ E_3$$

respectively, or more compactly, if i, k be reserved for 1, 2, 3 only,

$$F_{ik}=\mathbf{B};\ F_{i4}=\mathbf{E}. \tag{89a}$$

This then is the required correlation for the case in hand. Non-cartesian coördinates will be dealt with in the sequel.

Next, let $F^{\iota\kappa}$ be the supplement of $F_{\alpha\beta}$ defined, as in (34), by

$$F^{\iota\kappa}=g^{\iota\alpha}g^{\kappa\beta}F_{\alpha\beta}. \tag{90}$$

Then the group (B) of the electromagnetic equations will be replaced by the four equations

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\kappa}\left(\sqrt{g}\ F^{\iota\kappa}\right)=C^\iota, \tag{B_1}$$

where C^ι is a contravariant four-vector. Such also being the left hand member, *the divergence* of $F^{\iota\kappa}$, as in (47), the equations (B_1) will be generally contravariant. To compare them with (B) and to find the correlation proceed as before. Thus, on the one hand,

$$-\frac{\partial D_1}{\partial x_4}+\frac{\partial M_3}{\partial x_2}-\frac{\partial M_2}{\partial x_3}=\rho\frac{v_1}{c}, \text{ etc.}$$

$$\frac{\partial D_1}{\partial x_1}+\frac{\partial D_2}{\partial x_2}+\frac{\partial D_3}{\partial x_3}=\rho,$$

and on the other hand, remembering that $F^{\kappa\kappa}=0$ and $F^{\iota\kappa}=-F^{\kappa\iota}$,

$$-\frac{\partial}{\partial x_4}(\sqrt{-g}\,F^{41})+\frac{\partial}{\partial x_2}(\sqrt{-g}\,F^{12})-\frac{\partial}{\partial x_3}(\sqrt{-g}\,F^{31})=$$

$$\sqrt{-g}\,C^1, \text{ etc.},$$

$$\frac{\partial}{\partial x_1}(\sqrt{-g}\,F^{41})+\text{etc.}=\sqrt{-g}\,C^4.$$

The required correlation is, therefore,

$$\sqrt{-g}\,(F^{41},\ F^{42},\ F^{43})=D_1,\ D_2,\ D_3$$

$$\sqrt{-g}\,(F^{23},\ F^{31},\ F^{12})=M_1,\ M_2,\ M_3$$

or, in the previous abbreviated notation,

$$\sqrt{-g}\,F^{4i}=\mathbf{D};\ \sqrt{-g}\,F^{ik}=\mathbf{M}. \tag{89b}$$

Since $F^{\iota\kappa}$ is thus seen to embody the electric polarization and the magnetic force, it may be distinguished from its supplement by the name of the *electro-magnetic six-vector*. At the same time we have, by comparing the right-hand members of the two forms of equations,

$$\sqrt{-g}\,(C^1,\ C^2,\ C^3;\ C^4)=\rho\left(\frac{v_1}{c},\ \frac{v_2}{c},\ \frac{v_3}{c}\ ;\ 1\right)$$

or, more shortly,

$$C^i;\ C^4=\frac{\rho}{\sqrt{-g}}\left(\frac{\mathbf{v}}{c}\ ;\ 1\right), \tag{91}$$

exhibiting C^κ as the electric *four-current*. It is interesting to note that since we can put $v_i/c=dx_i/dx_4$ and $dx_4/dx_4=1$, the last correlation can also be written

$$C^\kappa=\frac{\rho}{\sqrt{-g}}\ \frac{dx_\kappa}{dx_4}. \tag{91'}$$

Since dx_κ is a contravariant vector as well as the four-current, the factor of dx_κ will be an invariant, and since $\sqrt{-g}\,dx_1\,dx_2\,dx_3\,dx_4$ is also an invariant, the volume of a world-element, we see that the electric charge $\delta e=\rho dx_1\,dx_2\,dx_3$ is again an invariant. Then, however, not ρ itself but ρ divided by the determinant $-|g_{ik}|$ will be the system-density of electricity.

It may be well to illustrate the general transformation formulae of $F_{\iota\kappa}$,

$$F'_{\iota\kappa} = \frac{\partial x_\alpha}{\partial x'_\iota} \frac{\partial x_\beta}{\partial x'_\kappa} F_{\alpha\beta},$$

by writing them out for the simplest case of two *inertial* systems S, S' in uniform translational motion relatively to each other. The transformation is in this case the familiar Lorentz transformation, *i.e.*, in cartesian coördinates and with the x_1 axis along the direction of motion,

$$x_1 = \gamma(x_1' + \beta x_4'),\ x_2 = x_2',\ x_3 = x_3',\ x_4 = \gamma(x_4' + \beta x_1'),$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$, if v be the velocity of S' relatively to S. First of all, since in this case the $g_{\iota\kappa}$ have their galilean values (in both systems), we have

$$\mathbf{B} = \mathbf{M},\ \ \mathbf{D} = \mathbf{E},$$

so that there is no need to consider the supplement of $F_{\iota\kappa}$; it is enough to treat $F_{\iota\kappa}$ itself. Next, since x_2, x_3 depend only on x_2', x_3', being equal to them respectively, we have

$$F'_{23} = M_1' = \frac{\partial x_\alpha}{\partial x_2'} \frac{\partial x_\beta}{\partial x_3'} F_{\alpha\beta} = F_{23} = M_1.$$

Similarly,

$$F'_{31} = \frac{\partial x_\alpha}{\partial x_1'} F_{3\alpha} = \gamma(F_{31} + \beta F_{34}),$$

i.e.,

$$M_2' = \gamma(\beta E_3 + M_2),$$

and so on. Thus we get the transformation formulae

$$M_1' = M_1,\ M_2' = \gamma(M_2 + \beta E_3),\ M_3' = \gamma(M_3 - \beta E_2)$$

$$E_1' = E_1,\ E_2' = \gamma(E_2 - \beta M_3),\ E_3' = \gamma(E_3 + \beta M_2),$$

familiar from the special relativity theory. The corresponding transformation of the four-current may be left as an exercise for the reader.

It will be kept in mind that the correlations of the forces, the polarizations and the current and charge density to the two conjugated six-vectors and the four-current given in (89*a*), (89*b*), (91) are valid only for the particular case of a cartesian or quasi-cartesian coördinate system. With other systems, such for instance as the polar coördinates, even in a galilean domain, the correlation formulae are more complicated, and contain besides the determinant g the several components $g_{\iota\kappa}$ of the metrical tensor or (in a non-galilean domain) parts of them, as will be seen later on. It is important

to understand that there is nothing general about these correlations, apart from the fact that $F_{\iota\kappa}$ embodies somehow the three-vectors **B** and **E**, and $F_{\iota\kappa}$ the vectors **D** and **M**, and C^{κ} the convection current and the charge density, everything being entangled with the metrical tensor and through it also with gravitation.

From the standpoint of general relativity the master equations are henceforth no more the broadened maxwellian equations (A), (B) but the set of generally covariant or contravariant equations (A_1), (B_1) with the metrical link (90) between the two six-vectors. It will be well to gather here these somewhat scattered equations; the whole generally covariant electromagnetic set is thus

$$\left.\begin{aligned} &\frac{\partial F_{\iota\kappa}}{\partial x_\lambda} + \frac{\partial F_{\kappa\lambda}}{\partial x_\iota} + \frac{\partial F_{\lambda\iota}}{\partial x_\kappa} = 0 \\ &\frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x_\kappa}\left(\sqrt{g}\,F^{\iota\kappa}\right) = C^{\iota} \\ &F^{\iota\kappa} = g^{\iota\alpha} g^{\kappa\beta} F_{\alpha\beta}\,. \end{aligned}\right\} \qquad \text{(IV)}$$

This will read as follows: the expansion of the magneto-electric six-vector vanishes; the divergence of the electro-magnetic six-vector, the supplement of the former, is equal to the electrical four-current.

36. *The four-potential.* Manifestly, the first of the equations (IV) will be identically satisfied if we put

$$F_{\iota\kappa} = \frac{\partial \phi_\iota}{\partial x_\kappa} - \frac{\partial \phi_\kappa}{\partial x_\iota}, \qquad (92)$$

where ϕ_ι is a covariant vector. If this be substituted, the six terms destroy themselves in pairs, and the covariant nature of ϕ_ι ensures the required tensor character of $F_{\iota\kappa}$, *the rotation* of ϕ_ι (cf. p. 61). The latter, which is seen to embody Maxwell's vector potential and the electrostatic potential, is called *the four-potential.*

With the correlation (89a) the six equations (92) become

$$\mathbf{B} = -\text{curl}\,(\phi_1, \phi_2, \phi_3), \ \mathbf{E} = \frac{\partial}{c\partial t}(\phi_1, \phi_2, \phi_3) - \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)\phi_4$$

or

$$\mathbf{B} = \text{curl}\,\mathbf{A}, \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{c\partial t} - \nabla\phi,$$

exhibiting the three-dimensional vector $\mathbf{A} = -(\phi_1, \phi_2, \phi_3)$ as Maxwell's vector-potential and $\phi = \phi_4$ as the electrostatic potential.

The first group of equations (IV) being thus satisfied by (92), the second group gives

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x}\left[\sqrt{g}\; g^{\iota\alpha} g^{\kappa\beta}\left(\frac{\partial\phi_\alpha}{\partial x_\beta} - \frac{\partial\phi_\beta}{\partial x_\alpha}\right)\right] = C^\iota, \tag{93}$$

which, assuming $g^{\iota\kappa}$ to be known, are four differential equations of the second order for as many components of the four-current. Since the four-potential enters only through its rotation, we can without loss to generality subject its components to a kind of solenoidal condition, as follows. If $\phi^\kappa = g^{\kappa\alpha}\phi_\alpha$ be the associated four-potential, a contravariant vector, then its *divergence* defined by (48) is a general invariant or scalar, and the condition in question can be written

$$\frac{\partial}{\partial x_\kappa}(\sqrt{g}\,\phi^\kappa) = 0. \tag{94}$$

In a galilean domain the equations (93), (94) become

$$\frac{\partial^2\mathbf{A}}{c^2\partial t^2} - \nabla^2\mathbf{A} = \frac{1}{c}\rho\mathbf{v}, \quad \frac{\partial^2\phi}{c^2\partial t^2} - \nabla^2\phi = \rho$$

$$\text{div}\,\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0,$$

the familiar equations of the electron theory for the vector potential $\mathbf{A} = -(\phi_1, \phi_2, \phi_3)$ and the electrostatic potential $\phi = \phi_4$. In general, however, the equations (93) for the four-potential will contain in a complicated way the components of the metrical tensor, which again means an entanglement of the electromagnetic with the gravitational field. This mutual relation of the two fields appears directly in the third of equations (IV) giving the general connection between the magneto-electric six-vector and its supplement.

Since the four-potential is a covariant and dx_κ a contravariant vector, their inner product

$$dl = \phi_\kappa \, dx_\kappa \tag{95}$$

is an invariant. This invariant *linear differential form* plays the same rôle with respect to electromagnetism as the quadratic differential form

$$ds^2 = g_{\iota\kappa} \, dx_\iota \, dx_\kappa$$

with respect to gravitation. As the latter determines, *inter alias*, the gravitational field, so does the former determine the electromagnetic field. This is only a different way of stating that the ϕ_κ, the coefficients of dl, determine the electromagnetic, similarly as the $g_{\iota\kappa}$ determine the gravitational field together with the riemannian metrical properties of space-time. Recently a differential geometry somewhat broader than Riemann's was proposed by Weyl who goes deep into the matter and attributes to the linear differential form an equally fundamental metrical (gauging) function as to the quadratic differential form. But reasons of space prevent us from entering here into this subject, and the interested reader must be referred to Weyl's own book* for further information. Moreover, these new physico-geometrical speculations, although undoubtedly attractive, are still being debated between Weyl and Einstein,† and may therefore be appropriately omitted in a book of the present type.

37. Let us once more return to the electromagnetic equations (A), (B) in order to compare them with the tensor equations (IV) for the case of a non-cartesian system of space coördinates. As a good example of this kind we may take any *orthogonal* curvilinear coördinates x_1, x_2, x_3. It is well known that if the space line-element in these coördinates be given by

$$d\sigma^2 = \frac{dx_1^2}{w_1^2} + \frac{dx_2^2}{w_2^2} + \frac{dx_3^2}{w_3^2} = \frac{dx_i^2}{w_i^2} \tag{96}$$

*H. Weyl, *Raum-Zeit-Materie*, 3rd ed., Berlin 1920, §16 and §34.

†Cf. Einstein's remarks to Weyl's paper, with Weyl's reply, in Berlin *Sitzungsber.*, 1918, and Einstein's recent paper, *ibidem*, 1921, pp. 261-264.

and if R_i be the components of a three-vector **R** tangential to the w_i-lines of the network,

$$\text{div}\,\mathbf{R} = w_1 w_2 w_3 \left[\frac{\partial}{\partial x_1}\left(\frac{R_1}{w_2 w_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{R_2}{w_3 w_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{R_3}{w_1 w_2}\right)\right] \tag{97}$$

and the curvilinear components of curl **R** are

$$(\text{curl}\,\mathbf{R})_1 = w_2 w_3\left[\frac{\partial}{\partial x_2}\left(\frac{R_3}{w_3}\right) - \frac{\partial}{\partial x_3}\left(\frac{R_2}{w_2}\right)\right], \text{ etc.} \tag{98}$$

With these expressions the group (A) of equations becomes, provided of course that the w_i are independent of time,

$$\frac{\partial}{\partial x_4}\left(\frac{B_1}{w_2 w_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{E_3}{w_3}\right) - \frac{\partial}{\partial x_3}\left(\frac{E_2}{w_2}\right) = 0, \text{ etc.,}$$

$$\frac{\partial}{\partial x_1}\left(\frac{B_1}{w_2 w_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{B_2}{w_3 w_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{B_3}{w_1 w_2}\right) = 0,$$

and similarly for the group (B) of electromagnetic equations. These equations are to be compared with the first and second of the tensor equations (IV). To find the required correlation in terms of the $g_{\iota\kappa}$ notice that if the domain is assumed to be a galilean one,* we have

$$ds^2 = g_{\iota\kappa}\, dx_\iota\, dx_\kappa = dx_4^2 - d\sigma^2,$$

so that

$$g_{ii} = -\frac{1}{w_i^2}, \quad g_{44} = 1,$$

and the remaining $g_{\iota\kappa}$ vanish. Under these circumstances the comparison gives at once, with g_i written for g_{ii},

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*Otherwise, say in the presence of gravitation, not the whole of $\frac{1}{w_i}$ is to be thrown upon the coefficient of dx_i in the expression for the length $d\sigma_i$ considered from the *system*-point of view.

$$\left.\begin{aligned}
&F_{23}=\sqrt{g_2\,g_3}\,B_1,\text{ etc.};\quad F_{41}=\sqrt{-g_1}\,E_1,\text{ etc.}\\
&F^{23}=\frac{1}{\sqrt{g_2\,g_3}}\,M_1,\text{etc.};\ F^{41}=\frac{1}{\sqrt{-g_1}}\,D_1,\text{ etc.}\\
&C^1=\frac{\rho v_1}{\sqrt{-g_1}}\,,\text{ etc., }C^4=\rho,
\end{aligned}\right\}\qquad(99)$$

which is the required correlation.

The relations between the polarizations and the forces, determined in general by the third of equations (IV), follow easily. In fact, since in the present case $g^{ii}=1/g_i$, $g^{44}=1$, and the remaining $g_{\iota\kappa}$ vanish, we have

$$F^{\iota\kappa}=\frac{1}{g_{\iota\iota}\,g_{\kappa\kappa}}\,F_{\iota\kappa}\,,$$

that is to say,

$$F^{ik}=\frac{1}{g_i\,g_k}\,F_{ik}\,,\quad F^{i4}=\frac{1}{g_i}\,F_{i4}\,,$$

and therefore, by (99), $B_1=M_1$, etc., and $E_1=D_1$, etc. In fine,

$$\mathbf{B}=\mathbf{M},\ \ \mathbf{D}=\mathbf{E},$$

the polarizations are identical with the forces, and the equations (A), (B) reduce to the usual electromagnetic equations for the vacuum, giving c as light velocity, and so on. This result might have been expected, for the present case differs from that corresponding to $ds^2=dx_4^2-dx_1^2-dx_2^2-dx_3^2$ solely by the use of curvilinear instead of cartesian coördinates.

38. Let us now consider the relation between **B, D** and **M, E** in another example which, besides being instructive in a general way, will show how the propagation of electromagnetic waves is influenced by gravitation.

If a system be used for which $g_{41}=g_{42}=g_{43}=0$, the four-dimensional line-element can be written

$$ds^2=g_{44}\,dx_4^2+g_{ik}\,dx_i\,dx_k\,,\quad i,k=1,2,3. \qquad (100)$$

In a weak gravitational field g_{44} as well as the g_{ik} will differ but little from their galilean values. Thus, if the x_i are

cartesian or quasi-cartesian coördinates, g_{44} and the g_{ii} will differ but little from $+1$ and -1 respectively, and the remaining g_{ik} will be small fractions. Thus, from the system-point of view,* the electromagnetic equations (A), (B) will be

$$\frac{\partial B_1}{\partial x_4} + \frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} = 0, \text{ etc.},$$

so that a comparison with the tensor equations will give again

$$F_{ik} = \mathbf{B},\ F_{i4} = \mathbf{E};\ F^{ik} = \frac{1}{\sqrt{-g}}\mathbf{M},\quad F^{4i} = \frac{1}{\sqrt{-g}}\mathbf{D}\,, \tag{89}$$

as in (89*a*), (89*b*).

Since $g_{4i} = 0$, the general relation $F_{\iota\kappa} = g_{\iota\alpha}\ g_{\kappa\beta}\ F^{\alpha\beta}$ between the two six-vectors will now give

$$F_{23} = F^{23}(g_{22}\,g_{33} - g_{23}^2) + F^{31}(g_{23}\,g_{31} - g_{21}\,g_{33}) + F^{12}(g_{21}\,g_{32} - g_{22}\,g_{31})$$

and two similar equations for F_{31}, F_{12}. But these are the solutions of the three equations

$$F^{23} = \frac{1}{h}\,(g_{11}\,F_{23} + g_{12}F_{31} + g_{13}F_{12}), \text{ etc.},$$

where h is the determinant $|g_{ik}|$. Now, $h = g/g_{44}$, and therefore

$$F^{23} = \frac{g_{44}}{g}\,(g_{11}F_{23} + g_{12}F_{31} + g_{13}F_{12}), \text{ etc.}$$

Again,

$$F_{41} = g_{4\alpha}\,g_{1\beta}\,F^{\alpha\beta} = g_{44}\,g_{1i}\,F^{4i}, \text{ etc.},$$

i.e.,

$$F_{41} = g_{44}(g_{11}F^{41} + g_{12}F^{42} + g_{13}F^{43})$$

and two similar equations for F_{42}, F_{43}. Whence, by (89),

$$M_1 = -\frac{g_{44}}{\sqrt{-g}}(g_{11}B_1 + g_{12}B_2 + g_{13}B_3), \text{ etc.}$$

$$E_1 = -\frac{g_{44}}{\sqrt{-g}}(g_{11}D_1 + g_{12}D_2 + g_{13}D_3), \text{ etc.},$$

*Analogously to the sense in which 'the system-velocity' of light was used previously, and contrasted with the *local* point of view.

or, solving for the polarisation components and noticing that $1/g_{44} = g^{44}$,

$$B_1 = -\sqrt{-g} \, . \, g^{44}(g^{11}M_1 + g^{12}M_2 + g^{13}M_3), \text{ etc.}$$

$$D_1 = -\sqrt{-g} \, . \, g^{44}(g^{11}E_1 + g^{12}E_2 + g^{13}E_3), \text{ etc.}$$

Thus **B** is exactly the same linear vector function of **M** as is **D** of **E**. Introducing the symmetrical linear vector operator

$$-\tilde{\omega} = \begin{vmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{vmatrix} \qquad (101)$$

we can write shortly

$$\mathbf{B} = \mu \mathbf{M}, \ \mathbf{D} = K\mathbf{E}, \qquad (102)$$

where

$$\mu = K = \sqrt{-g} \, . \, g^{44} \, \tilde{\omega}. \qquad (103)$$

In absence of gravitation the $g_{\iota\kappa}$ assume their galilean values, the operator $\tilde{\omega}$ becomes an idemfactor, $g^{44} = 1$, and $\mu = K = 1$, giving $\mathbf{B} = \mathbf{M}$ and $\mathbf{D} = \mathbf{E}$. From the system-point of view the vacuum is thus transformed by gravitation into a crystalline electromagnetic medium with anisotropic permeability μ and permittivity K. These operators have, however, by (103), at every point *common principal axes* (which are orthogonal) *and the same principal values.* Now, owing to this peculiarity the velocity of propagation of an electromagnetic wave, although varying from point to point and dependent upon the direction of the wave-normal, can be easily proved to be independent of the orientation of the light vector **D**. Thus, although the medium is anisotropic, there will be *no double refraction* due to the gravitational field.* In fact, if $\mathbf{n}$ be the wave-normal and $\mathfrak{v}$ the velocity, that is the system-velocity

*Cf. in this connection A. O. Rankine and L. Silberstein, *Propagation of light in a gravitational field*, Phil. Mag., vol. 39, 1920, p. 586.

of propagation, along the wave-normal, we have from the electromagnetic equations (A), (B), (102), with $\rho = 0$,

$$\frac{\mathfrak{v}}{c} K\mathbf{E} = V\mathbf{Mn}, \quad \frac{\mathfrak{v}}{c}\mu\mathbf{M} = V\mathbf{nE}, \tag{104}$$

as will be seen at once by considering a wave of discontinuity and using the general compatibility conditions given elsewhere.* Now, since the operator K is identical with μ, the last two equations give

$$\frac{\mathfrak{v}^2}{c^2} K\mathbf{E} + V\mathbf{n}(K^{-1}V\mathbf{nE}) = 0,$$

for every direction of $\mathbf{E}$. Here the operator K^{-1} is the inverse of K. If K_1, etc., be the principal values of K and n_1, etc., the components of $\mathbf{n}$ or the direction cosines of the wave-normal with respect to the principal axes, the last equation gives at once

$$\frac{\mathfrak{v}^2}{c^2} = \frac{\mathbf{n}K\mathbf{n}}{K_1 K_2 K_3} = \frac{K_1 n_1^2 + K_2 n_2^2 + K_3 n_3^2}{K_1 K_2 K_3},$$

i.e., a propagation velocity independent of the orientation of the light vector, which proves the statement.

If g_1, g_2, g_3 are the principal values of the vector operator $g_{11}, g_{12}, \ldots g_{33}$, the inverse of $-\tilde{\omega}$, then the principal values of $-\tilde{\omega}$ itself are $1/g_1$, etc., and we have, by (103) and since $g = g_1 g_2 g_3 g_{44}$,

$$\frac{\mathfrak{v}^2}{c^2} = -g_{44}\left[\frac{n_1^2}{g_1} + \frac{n_2^2}{g_2} + \frac{n_3^2}{g_3}\right]. \tag{105}$$

Such being the formula for the velocity of propagation on the electromagnetic theory of light, it is interesting to compare it with the light velocity v yielded directly by Einstein's fundamental equation $ds = 0$. This velocity is taken along '*the ray*' instead of the wave-normal. Thus, by (100), if $\mathbf{u}$ be a unit vector along the ray, and u_i its direction cosines,

*L. Silberstein, *Annalen der Physik*, vol. 26, 1908, p. 751 and vol. 29, 1909, p. 523, or *Theory of Relativity*, London, 1914, p. 56.

$$g_{44}\frac{c^2}{v^2} = -g_{ik}\frac{dx_i}{d\sigma}\frac{dx_k}{d\sigma} = -g_{ik}\,u_i\,u_k\,,$$

and especially if u_i be the direction cosines with respect to the principal axes of the operator $g_{11}, g_{12}, \ldots g_{33}$,

$$\frac{c^2}{v^2} = -\frac{1}{g_{44}}\left[g_1\,u_1^2+g_2\,u_2^2+g_3\,u_3^2\right]. \tag{106}$$

Formula (85), used in connection with the bending of rays around the sun, is only a special case of (106). In that case the principal axes are along the radial and all the transversal directions, while the principal values are

$$g_1 = -\frac{1}{g_{44}} = -\frac{1}{g_4}\,,\ g_2 = g_3 = -1,$$

and $u_1^2 = \cos^2\eta$, $u_2^2+u_3^2 = \sin^2\eta$, so that (106) reduces to (85).

If the wave-normal $\mathbf{n}$ coincides with a principal axis, say with the first one, we have, by (105), $\mathfrak{v}^2/c^2 = -g_{44}/g_1$, and by (106), $c^2/v^2 = -g_1/g_{44}$; that is to say, $v = \mathfrak{v}$, as it should be. For then the ray falls into the wave-normal. But in general the ray does not coincide with the wave-normal, and so does v differ from $\mathfrak{v}$. The question whether the null-line equation (106) is always compatible with the electromagnetic equation (105) may be left to the care of the reader. If the ray be defined, as usual, by the Poynting flux of energy, its direction will be that of the vector product $V\mathbf{EM}$, and all questions concerning the light ray will follow from (104) with $K = \mu$ as given by (103).

39. *Ponderomotive force, and energy tensor of the electromagnetic field.* The general tensors corresponding to these were easily suggested by the results already known from the special relativity theory.

The inner product of the magneto-electric six-vector and the four-current, *i.e.*, the covariant vector

$$P_\iota = F_{\iota\kappa}\,C^\kappa, \tag{V}$$

gives the ponderomotive force on a charge, per unit volume, together with its activity or, in other words, the momentum and the energy transferred, per unit volume and unit time, from the electromagnetic field upon the electric charges.

In fact, using for instance cartesian coördinates and $g=-1$, we have for the first three components of P_ι, by (89) and (91),

$$P_1=\rho\left[\frac{1}{c}(v_2B_3-v_3B_2)+E_1\right], \text{ etc.},$$

or if P_1, P_2, P_3 be condensed into the three-vector $\mathbf{P}$,

$$\mathbf{P}=\rho\left[\mathbf{E}+\frac{1}{c}\ V\mathbf{v}\,\mathbf{B}\right]$$

which is the familiar formula for the ponderomotive force, while the fourth component becomes

$$P_4=-\frac{\rho}{c}(E_1v_1+E_2v_2+E_3v_3)=-\frac{\rho}{c}(\mathbf{E}\mathbf{v})$$

or, since $V\mathbf{v}\mathbf{B}=0$,

$$P_4=-\frac{1}{c}(\mathbf{P}\mathbf{v})$$

which, apart from the factor $-1/c$, is the activity of $\mathbf{P}$. Somewhat more generally, the same formulae will hold with ρ replaced by $\rho/\sqrt{-g}$.

But it will be understood that from the standpoint of general relativity the master formula for the electromagnetic momentum and energy transfer is again (V), as were before the electromagnetic field equations, all generally covariant.

By means of (IV) and (V) the four-force P_ι can be represented as the covariant derivative of a second rank tensor, a generalization of the array of maxwellian stresses, momentum and energy density. Following Einstein's example it will be enough to give here the required form of P_ι for such coördinates for which $g=-1$, and therefore, by the second of (IV),

$$C^\kappa=\frac{\partial F^{\kappa\lambda}}{\partial x_\lambda}.$$

Thus, by (V),

$$P_\iota=\frac{\partial}{\partial x_\lambda}(F_{\iota\kappa}F^{\kappa\lambda})-F^{\kappa\lambda}\frac{\partial F_{\iota\kappa}}{\partial x_\lambda}.$$

The second term is, by the first of the equations (IV),

$$F^{\kappa\lambda}\frac{\partial F_{\iota\kappa}}{\partial x_\lambda} = -F^{\kappa\lambda}\left(\frac{\partial F_{\kappa\lambda}}{\partial x_\iota} + \frac{\partial F_{\lambda\iota}}{\partial x_\kappa}\right)$$

$$= -\tfrac{1}{2}F^{\kappa\lambda}\frac{\partial F_{\kappa\lambda}}{\partial x_\iota} - \left[\tfrac{1}{2}F^{\kappa\lambda}\frac{\partial F_{\kappa\lambda}}{\partial x_\iota} + F^{\kappa\lambda}\frac{\partial F_{\lambda\iota}}{\partial x_\kappa}\right].$$

But the bracketed expression vanishes. In fact, since the summation is to be extended over all κ, λ, and since both F-tensors are antisymmetric, this expression can be written

$$F^{\kappa\lambda}\left(\frac{\partial F_{\kappa\lambda}}{\partial x_\iota} + \frac{\partial F_{\lambda\iota}}{\partial x_\kappa} - \frac{\partial F_{\kappa\iota}}{\partial x_\lambda}\right)$$

to be summed only over $\kappa < \lambda$. But the third term of the bracketed factor is $+\partial F_{\iota\kappa}/\partial x_\lambda$, so that the whole factor of $F_{\kappa\lambda}$ vanishes, by the first of (IV). Thus

$$F^{\kappa\lambda}\frac{\partial F_{\iota\kappa}}{\partial x_\lambda} = -\tfrac{1}{2}F^{\kappa\lambda}\frac{\partial F_{\kappa\lambda}}{\partial x_\iota} = -\tfrac{1}{2}g^{\kappa\alpha}g^{\lambda\beta}F_{\alpha\beta}\frac{\partial F_{\kappa\lambda}}{\partial x_\iota},$$

and since here κ, λ can be replaced by α, β and *vice versa*,

$$F^{\kappa\lambda}\frac{\partial F_{\iota\kappa}}{\partial x_\lambda} = -\tfrac{1}{4}g^{\kappa\alpha}g^{\lambda\beta}\frac{\partial}{\partial x_\iota}(F_{\alpha\beta}F_{\kappa\lambda})$$

$$= -\tfrac{1}{4}\frac{\partial}{\partial x_\iota}(F_{\kappa\lambda}F^{\kappa\lambda}) + \tfrac{1}{4}F_{\alpha\beta}F_{\kappa\lambda}\frac{\partial}{\partial x_\iota}(g^{\kappa\alpha}g^{\lambda\beta}).$$

The last term can be transformed into $-\tfrac{1}{2}F^{\kappa\tau}F_{\kappa\lambda}g^{\lambda\rho}\partial g_{\rho\tau}/\partial x_\iota$, so that

$$P_\iota = \frac{\partial}{\partial x_\lambda}(F_{\iota\kappa}F^{\kappa\lambda}) - \tfrac{1}{4}\frac{\partial}{\partial x_\iota}(F_{\kappa\lambda}F^{\kappa\lambda}) - \tfrac{1}{2}F_{\kappa\lambda}F^{\kappa\tau}g^{\lambda\rho}\frac{\partial g_{\rho\tau}}{\partial x_\iota}.$$

Finally, if we denote by F the invariant $F_{\kappa\lambda}F^{\kappa\lambda}$ and introduce the mixed tensor

$$T_\kappa^\lambda = \tfrac{1}{4}F\delta_\kappa^\lambda - F_{\kappa\alpha}F^{\lambda\alpha}, \tag{107}$$

the last formula becomes

$$P_\iota = \frac{\partial T_\iota^\alpha}{\partial x_\alpha} - \tfrac{1}{2}g^{\kappa\lambda}\frac{\partial g_{\kappa\nu}}{\partial x_\iota}T_\lambda^\nu, \tag{108}$$

exhibiting the four-force in terms of T_ι^κ, *the energy-tensor of the electromagnetic field.*

To recognize in the latter an old friend consider a galilean domain and use cartesian coördinates. Then, the $g_{\iota\kappa}$ being constant, (108) reduces to the familiar equation

$$P_\iota = \frac{\partial T_\iota^\alpha}{\partial x_\alpha},$$

and since, by (89), in the present case, $F_{23} = F^{23} = M_1$, etc., $F_{14} = -F^{14} = E_1$, etc., we have

$$F = F_{\alpha\beta}F^{\alpha\beta} = 2\,(M^2 - E^2),$$

and (107) gives

$$T_1^1 = (E_1^2 - \tfrac{1}{2}E^2) + (M_1^2 - \tfrac{1}{2}M^2),$$

$$T_1^2 = T_2^1 = E_1E_2 + M_1M_2, \text{ etc.},$$

which are Maxwell's electromagnetic stress components,* further

$$T_1^4 = T_4^1 = -(E_2M_3 - E_3M_2), \text{ etc.},$$

which are the negative components of the energy flux divided by c, or the components of electromagnetic momentum per unit volume, and finally

$$T_4^4 = -\tfrac{1}{2}(E^2 + M^2),$$

which is the negatived density of electromagnetic energy.

The right hand member of (108) can be shown to be the divergence of the mixed tensor T_ι^κ or its contracted covariant derivative $T_{\iota\alpha}^\alpha$ as defined by (47*a*). In fact, since for constant g, by (67), $\left\{ {\alpha\beta \atop \alpha} \right\} = 0$, the said divergence reduces to

$$T_{\iota\alpha}^\alpha = \frac{\partial T_\iota^\alpha}{\partial x_\alpha} - \left\{ {\iota\alpha \atop \beta} \right\} T_\beta^\alpha, \tag{42b}$$

and since in our case T_β^α is symmetrical, this can be shown to be identical with

$$T_{\iota\alpha}^\alpha = \frac{\partial T_\iota^\alpha}{\partial x_\alpha} + \frac{1}{2}\frac{\partial g^{\kappa\lambda}}{\partial x_\iota} T_{\kappa\lambda}, \tag{42c}$$

where $T_{\kappa\lambda} = g_{\kappa\nu}\, T_\lambda^\nu$. On the other hand, since $g^{\kappa\lambda} g_{\kappa\nu}$ is a constant, to wit δ_ν^λ, we have

$$g^{\kappa\lambda}\frac{\partial g_{\kappa\nu}}{\partial x_\iota} = -g_{\kappa\nu}\frac{\partial g^{\kappa\lambda}}{\partial x_\iota}$$

[a relation to be used also in passing from (42*b*) to (42*c*)], and

***Tensions* proper being counted positive.

the second term of (108) becomes identical with the second term of (42c). Thus

$$P_\iota = T^\alpha_{\iota\alpha} \equiv \text{Div}\ (T^\kappa_\iota) \qquad (108a)$$

exhibiting the four-force as *the divergence* of the mixed energy-tensor of the electromagnetic field.

If the electric charges are under the exclusive control of the electromagnetic field, the total four-force P_ι vanishes, and we have

$$T^\alpha_{\iota\alpha} \equiv \text{Div}\ (T^\kappa_\iota) = 0. \qquad (109)$$

These four equations are perfectly analogous to the 'equations of matter', (65), given in Chapetr IV, the 'tensor of matter' being now replaced by the energy-tensor of the electromagnetic field defined in (107). These equations express in either case the principles of energy and of momentum.

Instead of the mixed tensor (107) we can introduce the covariant electromagnetic tensor $g_{\iota\nu} T^\nu_\kappa = T_{\iota\kappa}$. If the form (III) of the gravitational field-equations be used, then in the presence of an electromagnetic field the components of the latter tensor (multiplied by the gravitation constant) have to be included in the corresponding components of the tensor of matter appearing in the right-hand member of those equations. Thus both kinds of stresses, energy, etc., contribute to the curvature tensor $G_{\iota\kappa}$ and through it codetermine the gravitational field. The contributions of the electromagnetic tensor components are, of course, for all technically obtainable fields, exceedingly small as compared with those due to matter in the narrower sense of the word. Theoretically, however, the rôles of the two kinds of energy-tensors are equivalent.

APPENDIX.

A. Manifolds of Constant Curvature.

As was mentioned in Chapter III, an n-dimensional manifold of constant isotropic riemannian curvature K, positive, nil or negative, is characterized by the differential equations (54), which can be deduced from the general formula (53) for the riemannian curvature.* If we put

$$K = \frac{1}{R^2},$$

where R may be any constant, imaginary or real, finite or infinite, the said equations are

$$(\iota\lambda, \mu\kappa) = \frac{1}{R^2}(g_{\iota\mu} g_{\lambda\kappa} - g_{\iota\kappa} g_{\lambda\mu}), \tag{110}$$

to be satisfied for all $\iota, \lambda, \kappa, \mu$. In order to pass from Riemann's covariant symbols to the mixed curvature tensor use (50a). Thus, multiplying both sides of (110) by $g^{\lambda a}$ and taking account of (32),

$$B^{a}_{\iota\kappa\mu} = \frac{1}{R^2}(\delta^{a}_{\kappa} g_{\iota\mu} - \delta^{a}_{\mu} g_{\iota\kappa}).$$

Einstein's tensor $G_{\iota\kappa}$ is the contracted curvature tensor $B^{a}_{\iota\kappa a}$, *i.e.*,

$$G_{\iota\kappa} = \frac{1}{R^2}(\delta^{a}_{\kappa} g_{\iota a} - \delta^{a}_{a} g_{\iota\kappa}).$$

The first term in the brackets is simply $g_{\iota\kappa}$, while the second,

*Cf. also W. Killing, *Die Nicht-Euklidischen Raumformen*, Leipzig 1885, Section 123.

in which δ_α^α or 1 is to be taken n times, is equal $ng_{\iota\kappa}$. Thus, for a manifold of *n dimensions*, of the said kind,

$$G_{\iota\kappa} = -\frac{n-1}{R^2} g_{\iota\kappa}, \tag{111}$$

for all values of ι, κ. In fine, the contracted curvature tensor is proportional to the metrical tensor $g_{\iota\kappa}$. For a three-space the constant factor is $-2/R^2$, and for a four-manifold $-3/R^2$, and so on. Notice that we are dealing here with *isotropic* manifolds,—a remark which will be of importance in the sequel.

The curvature invariant is $G = g^{\iota\kappa} G_{\iota\kappa}$, and since $g^{\iota\kappa} g_{\iota\kappa} = n$, we have, by (111),

$$G = -\frac{n(n-1)}{R^2}. \tag{112}$$

This justifies, in general, the name of 'mean curvature' mentioned in Chapter IV and given to G by some authors. For a three-space we have

$$G = -\frac{6}{R^2}, \tag{112^3}$$

and for a four-fold, provided always it were isotropic, we should have

$$G = -\frac{12}{R^2}. \tag{112^4}$$

It was known for a long time that the line-element of a three-space of constant curvature $1/R^2$ is, in polar coördinates x_1, x_2, $x_3 = r$, ϕ, θ,

$$d\sigma^2 = dr^2 + R^2 \sin^2\frac{r}{R} \cdot [d\phi^2 + \sin^2 \phi d\theta^2]. \tag{113}$$

In fact, availing himself of (75), the reader will find for (113), as the only surviving components,

$$G_{11} = -\frac{2}{R_2},\ G_{22} = -2\sin^2\frac{r}{R},\ G_{33} = -\sin^2\phi\, G_{22},$$

that is to say,

$$G_{ii} = -\frac{2}{R^2} g_{ii}, \tag{111$_3$}$$

thus verifying (111) for the case $n=3$, whence also $G=-6/R^2$, as above.

Manifestly, if we took for $d\sigma^2$ the negative of (113), or inverted the signs of all g_{ik}, we should have $G_{ii}=+\frac{2}{R^2}g_{ii}$. Now, it will be well to notice that the same is the case if we subtract the (113)-value of $d\sigma^2$ from the squared differential of a fourth coördinate multiplied by a constant; that is to say, for a four-dimensional manifold defined by

$$ds^2=dx_4^2-dr^2-R^2\sin^2\frac{r}{R}\cdot[d\phi^2+\sin^2\phi d\theta^2] \tag{114}$$

we have [not (111) with $n=4$ but]

$$G_{ii}=\frac{2}{R^2}\,(i=1,2,3);\quad G_{44}=0,$$

as the reader can verify explicitly, and therefore,

$$G=\frac{1}{g_{ii}}G_{ii}=+\frac{6}{R^2}.$$

In fine, for a *four-fold*, say space-time, of the type (114) the three curvature components and the invariant G have the same values as for an isotropic *three*-space with changed signs. Notice that this result does by no means clash with the general equations (111) and (112). For the space-time determined by the line-element (114) is *not* isotropic with respect to its riemannian curvature, even if x_4 be replaced by $\sqrt{-1}\,x_4$.

The latter line-element plays an important rôle in Einstein's recently modified theory of which a brief account will be given in Appendix, **B**.

Consider the four-fold defined by the somewhat more general line-element

$$ds^2=g_4dx_4^2-dr^2-R^2\sin^2\frac{r}{R}[d\phi^2+\sin^2\phi d\theta^2],$$

where g_4, written for g_{44}, is a function of r alone. Then, with $h_4=\log g_4$, the only surviving G-components will be

$$\left.\begin{aligned} G_{11} &= \tfrac{1}{4}\,(h'^2_4+2h_4'') - \frac{2}{R^2} \\ G_{22} &= \frac{1}{\sin^2\phi}\,G_{33} = \frac{Rh'_4}{4}\;\sin\frac{2r}{R} - 2\sin^2\frac{r}{R} \\ \frac{1}{g_4}\,G_{44} &= -\tfrac{1}{4}\,(h'^2_4+2h_4'') - \frac{h'_4}{R}\cot\frac{r}{R} \end{aligned}\right\} \qquad (115)$$

whence the curvature invariant $G = -G_{11}+2G_{22}/g_2+G_{44}/g_4$, with $g_2 = -R^2\sin^2(r/R)$,

$$G = \frac{6}{R^2} - \tfrac{1}{2}\,(h'^2_4+2h_4'') - \frac{2h'_4}{R}\cot\frac{r}{R}\,.$$

Let us now require that G should be *constant* (which is, at any rate a necessary condition for $G_{\iota\kappa} : g_{\iota\kappa} = \text{const.}$). Then the last formula will be a differential equation for $h_4 = \log g_4$. Now, this equation can be satisfied by

$$g_4 = \cos^2 ar,$$

where a is a constant. In fact, this assumption gives

$$h'_4 = -2a\tan ar,\;\; h''_4 = -2a^2/\cos^2 ar$$

and reduces the last equation to

$$2a^2 + \frac{4a}{R}\cot\frac{r}{R}\,.\,\tan ar = G - \frac{6}{R^2} = \text{const.},$$

and this equation can only be satisfied either by $a=0$, *i.e.*, $g_4=1$, and

$$G = \frac{6}{R^2},$$

which leads to the line-element just considered, or by $a=1/R$, *i.e.*, $g_4=\cos^2(r/R)$, and $G=12/R^2$, which gives the line-element

$$ds^2 = \cos^2\frac{r}{R}\,.\,dx_4^2 - dr^2 - R^2\sin^2\frac{r}{R}\,[d\phi^2+\sin^2\phi\, d\theta^2], \qquad (116)$$

utilized by de Sitter. (Cf. Appendix, **C**, *infra*). The con-

stant value of the invariant G is in this case

$$G = \frac{12}{R^2},$$

that is to say, apart from the changed sign, such as would correspond, by (112), to a genuine isotropic *four*-fold considered at the beginning. Moreover, introducing $g_4 = \cos^2(r/R)$ into (115), we have at once

$$G_{11} = -\frac{3}{R^2}\ , \quad G_{22} = \frac{1}{\sin^2\phi} G_{33} = -3\sin^2\frac{r}{R}\ , \quad G_{44} = \frac{3}{R^2} g_4\ ,$$

and since $g_1 = -1$, $g_2 = -R^2 \sin^2(r/R)$, and all components with $\iota \neq \kappa$ vanish,

$$G_{\iota\kappa} = \frac{3}{R^2} g_{\iota\kappa}\ . \tag{116^1}$$

which, apart from the changed sign of the constant factor, agrees with (111) for $n = 4$.

On the other hand, substituting into (115) the alternative solution $g_4 = 1$ we have, for the line-element (114),

$$G_{ii} = \frac{2}{R^2} g_{ii}\ (i = 1, 2, 3)\ ;\quad G_{44} = 0. \tag{114^1}$$

The best way of stating the properties of the two solutions is to write the corresponding contravariant tensors which in our case reduce to $G^{\iota\iota} = G_{\iota\iota}/g_{\iota\iota}$. These are, for the line-element (114),

$$G^{11} = G^{22} = G^{33} = \frac{2}{R^2}\ , \quad G^{44} = 0, \tag{114^2}$$

and for the line-element (116),

$$G^{11} = G^{22} = G^{33} = G^{44} = \frac{3}{R^2}\,. \tag{116^2}$$

Thus the time-space defined by the line-element (116) behaves, apart from the common sign change, as an orderly four-fold of constant and *isotropic* riemannian curvature. This is its characteristic difference from the manifold defined by (114) which is deprived of isotropy and is a rather loose, uneven *mélange* of time and space. Such at least would be

the comparison of Einstein's line-element (114) with de Sitter's, (116), from the standpoint of general geometry. Their physical merit must, of course, be judged by other standards.

B. Einstein's New Field-Equations and Elliptic Space.

About two years after the publication of the original form of the gravitational field-equations, (III), Chapter IV, Einstein found weighty reasons for slightly modifying them.* Without attempting an exhaustive discussion of all his reasons for that change or amplification we shall give here a brief account of his new field-equations and of some of their consequences.

The tensor of matter $T_{\iota\kappa}$ being given, the metrical and at the same time the gravitation tensor components $g_{\iota\kappa}$ are not, of course, determined by the field-equations alone, as indeed would be the case with any other set of differential equations in infinite space (and time). A necessary supplement of the data consisted, exactly as in the case of Laplace-Poisson's equation, in prescribing the behaviour of the $g_{\iota\kappa}$ at infinity. Now, as may best be seen from the example of the radially symmetrical field treated in Chapter V, the $g_{\iota\kappa}$ were assumed to tend 'at infinity', that is, for ever growing r/L, to their *galilean* values $\bar{g}_{\iota\kappa}$, say in cartesian coördinates,

$$\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

But such boundary or limit conditions, not being independent of the choice of the coördinate system, have seemed 'repugnant to the spirit of the relativity principle'. In fact, to remain generally invariant the limit tensor would have to be an array of sixteen zeros. Moreover, the adoption of the galilean or inertial tensor at infinity would be tantamount to giving up the requirement of the relativity of inertia. For whereas the inertia or mass of a particle generally depends upon the

*A. Einstein, *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie.* Berlin *Sitzungsberichte*, 1917, pp. 142-152.

$g_{\iota\kappa}$ and these are even at the surface of the sun but slightly different from $\bar{g}_{\iota\kappa}$, the mass of the particle at infinity would differ but very little from what it is near the sun or other celestial giants. In fine, the bulk of its mass would be independent of other bodies, and if the particle existed alone in the whole universe, it would still retain practically all its mass. As a matter of fact we do not know whether such would not be the case.* But somehow, not uninfluenced by Mach's older ideas, Einstein inclines to the belief that every particle owes its whole inertia to all the remaining matter in the universe. Yet another reason against the said conditions at infinity is given which is based on considerations borrowed from the statistical theory of gases and which would equally apply to Newton's theory. But for this the reader must be referred to Einstein's original paper (*l. c.*, §1).

In conclusion Einstein confesses his inability to build up any satisfactory conditions at infinity, in space that is.† But here a way out naturally suggested itself. The conditions at infinity being hard or perhaps impossible to find, let the world or universe be *closed in all its space extensions.* If this be a possible assumption, no such conditions were needed.

Thus Einstein comes to assume space to be a finite, closed three-fold of constant curvature, in short an *elliptic space*, either of the antipodal (spherical) or of the polar, properly 'elliptic', kind. But, as we saw before, the curvature properties of space-time are modified by the presence of matter, the invariant G, for instance, being proportional to the density of matter. Thus the curvature of space, as a section of the four-fold, can only be approximately constant and isotropic, and Einstein assumes therefore that space is elliptic or very nearly so *on the whole*, deviating here and there, within and near condensed matter, from the average value of its curvature $1/R^2$ and from isotropy, somewhat as, in two dimensions, a

*Provided, of course, we had some massless phantoms to serve us as a reference system and thus to enable us to state the lonely particle's perseverance in uniform motion.

†'Für das räumlich Unendliche'. There is nowhere a mention of the behaviour at infinite past or future, no doubt, because such questions with regard to time are not urgent in the usual (stationary) type of problems.

slightly corrugated or wrinkled sphere. As we know, the line-element of such a three-space is

$$d\sigma^2 = dr^2 + R^2 \sin^2 \frac{r}{R} (d\phi^2 + \sin^2\phi d\theta^2),$$

and Einstein constructs the line-element which is to determine the four-world 'on the whole' by simply subtracting $d\sigma^2$ from $dx_4^2 = c^2dt^2$.

In short, far enough from condensed matter, stars, planets, and so on, his line-element, in polar coördinates, is

$$ds^2 = dx_4^2 - dr^2 - R^2 \sin^2 \frac{r}{R} (d\phi^2 + \sin^2\phi d\theta^2), \tag{114}$$

a differential form treated in Appendix **A**.*

Now, this line-element is incompatible with Einstein's older field-equations (III). In fact, the corresponding curvature tensor consists of the only surviving components

$$G_{ii} = \frac{2}{R^2} g_{ii}\,, \quad G_{44} = 0; \quad G = \frac{6}{R^2}\,, \tag{114^1}$$

*From the four-dimensional point of view, the assumption that three-dimensional space is elliptic is, of course, as unsatisfactory as the older assumption of galilean $g_{\iota\kappa}$ at infinity. For although the space properties as defined by $d\sigma^2$ are invariant for transformations of the x_1, x_2, x_3 alone into any x'_1, x'_2, x'_3, they cease to be so when all four coördinates are freely transformed. What is then invariant are the curvature properties of the four-fold of which the three-space is an arbitrary section. If at least the four-fold (114) were isotropic, Einstein's elliptic space could be invariantly defined as that of its sections to which corresponds the *minimum* mean curvature, and this is the mean curvature of the four-fold itself (cf. W. Killing, *loc. cit.*, pp. 79-83). But the four-fold defined by (114) is by no means isotropic, as was explained in **A.** Figuratively, and with some licence, it resembles not a sphere but rather the surface of a circular cylinder. By (114) not only the value of the curvature of three-space remains unsettled but even its property of being at all a closed space. In fine, the assumption that three-space is elliptic should be as 'repugnant to the spirit of relativity' as was the older condition at infinity. But as a matter of fact it did not appear to Einstein in that light.

The clearest way of stating Einstein's new assumption is to say that, outside of condensed matter, it is possible to choose a coördinate system in which the line-element ds^2 assumes the form (114).

and if these values be introduced into the field-equations (IIIa), which are identical with (III), the result is

$$\frac{1}{R^2} g_{ii} = \frac{8\pi}{c^2} T_{ii}, \quad \frac{3}{R^2} = \frac{8\pi}{c^2} T_{44}.$$

But 'on the whole', that is, outside of condensed matter, T_{11}, T_{22}, T_{33} are to vanish (though the value of T_{44} and $T = \rho$ need not be prejudiced), and since actually $g_{11} = -1$, etc., the incompatibility of (114) with (III) is manifest.

Such being the case, Einstein is driven to modify his original equations (III) by subtracting from their left-hand members the terms $\lambda g_{\iota\kappa}$ with a constant λ. Thus his new field-equations are

$$G_{\iota\kappa} - \lambda g_{\iota\kappa} = -\frac{8\pi}{c^2}(T_{\iota\kappa} - \tfrac{1}{2} g_{\iota\kappa} T), \tag{117}$$

and since these give, obviously,

$$G - 4\lambda = \frac{8\pi}{c^2} T, \tag{117a}$$

they can also be written

$$G_{\iota\kappa} - \tfrac{1}{2}(G - 2\lambda) g_{\iota\kappa} = -\frac{8\pi}{c^2} T_{\iota\kappa}. \tag{117b}$$

Since the supplementary term $\lambda g_{\iota\kappa}$ is itself covariant of rank two, the general covariance of the new equations is manifest.

It remains to evaluate the constant λ in terms of the curvature $1/R^2$. Now, if we assumed that outside of 'condensed matter' there is no matter at all, *i.e.*, $T_{\iota\kappa} = 0$ for all ι, κ, we should have, by (114^1) and the first of (117b), $\lambda = 1/R^2$, clashing with (117a), through (114^1) which would require $\lambda = 3/R^2$. But, as Einstein expressly states, his new theory is to be associated with the approximate concept that all the matter of the universe is *spread uniformly over immense spaces*. In other words, Einstein substitutes for the granular structure of the universe (the grains being not only planets but stars, nebulae and similar giants) a macroscopically homogeneous distribution of matter, exactly as for many purposes a con-

tinuous homogeneous medium is substituted for an assemblage of molecules or atoms. The total mass contained in the universe being M and its volume V, Einstein's homogeneous density, prevailing on the whole, is

$$\rho_0 = \frac{M}{V} .$$

Only here and there, within the celestial bodies, the density ρ exceeds ρ_0 considerably, and is perhaps somewhat larger in interstellar spaces within our galaxy than half way between one star cluster or 'island universe' and another, a million or more light years apart. Moreover, basing himself upon the known fact of the small relative velocity of stars as compared with the light velocity, Einstein makes the approximate assumption that there is a coördinate system, relatively to which matter is on the whole permanently at rest, and in which therefore the tensor of matter is reduced to its 44-th component which is then also its invariant $T = \rho$.

In fine, we have outside of condensed matter

$$T_{44} = T = \rho_0$$

as the only surviving component, and therefore, by (114¹) and (117b),

$$\lambda = \frac{1}{R^2} = \frac{4\pi}{c^2}\rho_0.$$

Thus, Einstein's new field-equations (117) become ultimately

$$G_{\iota\kappa} - \tfrac{1}{2}\left(G - \frac{2}{R^2}\right) g_{\iota\kappa} = -\frac{8\pi}{c^2} T_{\iota\kappa} . \tag{117c}$$

At the same time we see that the curvature of space on the whole is proportional to the average density of matter,

$$\frac{1}{R^2} = \frac{4\pi}{c^2}\rho_0 . \tag{118}$$

The whole volume of elliptic space of the *polar* or properly *elliptic* kind being

$$V = \pi^2 R^3,$$

the total mass of the universe, in astronomical units, will be

$$M=\frac{\pi c^2}{4}R, \tag{119}$$

which moved some authors to the enthusiastic exclamation: 'the more matter, the more room'. The corresponding 'gravitation radius', or better, the mass in *bary-optical* units, which is a *length*, would be

$$L=\frac{M}{c^2}=\frac{\pi R}{4}, \tag{119a}$$

or just one-quarter of the total length of an elliptic straight line.*

According even to our coarse knowledge of the average density of matter (some thousand suns per cubic parsec), and in view of the formula (118), it is impossible to believe in a curvature radius much smaller than 10^{12} astronomical units or, say, $R=10^{20}$ kilometers. This would mean, by (119a), a total mass amounting again, in bary-optical units, to almost 10^{20} kilometers. To this tremendous total our own sun contributes but $1\frac{1}{2}$ kilometers, and our whole galaxy not more than 10^{10} kilometers. The total would thus require 10^{10} such galaxies or Shapley's 'island universes'. All these stellar systems may perhaps be found among the spirals. But if Shapley's estimate (Bull. Nat. Res. Council, 1921, No. 11, *The Scale of the Universe*) be materially correct, these island universes are from 500 thousand to 10 million light years apart, and then it remains to be seen whether the last mentioned space would be ample enough. Yet it would certainly be foolish to deny the possibility of a much larger R and of the existence of many more island universes. That Einstein's requirement, at least in the present state of astronomical knowledge, can at any rate be satisfied, is perhaps best seen from its form (118) which is compatible with as small an average density as we please.

*The total length of a straight line (geodesic) in the polar kind of space is πR, and in the antipodal or spherical kind of space $2\pi R$. The total volume of the latter space is $2\pi^2 R^3$, which would give the double mass, as in Einstein's paper. The space in question being thus far defined only differentially, the choice between the polar and antipodal kind remains free.

Further details concerning these cosmological speculations will be found in de Sitter's third paper on Einstein's Theory of Gravitation,* where the rôle played by elliptic space in astronomy since the time of Schwarzschild (1900) is discussed.

The light rays corresponding to Einstein's line-element (114) turn out to be straight lines in elliptic space, and these lines, described with uniform velocity, are also the orbits of free particles. Planetary motion would undergo some modifications due to the finite value of R; but these are, for the present, too small to be detected. Nor does Einstein's 'cosmological term', as the supplement $g_{\iota\kappa}/R^2$ to his original field-equations is called, lead to any other predictions verifiable in our days by experiment or observation.

C. Space-Time according to de Sitter.

Returning to Einstein's amplified field-equations (117) let us assume, with de Sitter, that there is outside of 'condensed matter' no matter at all, so that in such domains all the components of $T_{\iota\kappa}$, including T_{44}, vanish. Thus we shall have, in free space, so to speak,

$$G_{\iota\kappa} = \lambda g_{\iota\kappa}$$

for all ι, κ. Now, as we saw in Appendix **B**, these equations, which are of the form of (111), can be satisfied by the line-element (116), and give $G = 12/R^2$. And since, on the other hand,

$$G = g^{\iota\kappa} G_{\iota\kappa} = \lambda g^{\iota\kappa} g_{\iota\kappa} = 4\lambda,$$

we have

$$\lambda = \frac{3}{R^2}.$$

This is the solution of the cosmological problem proposed by de Sitter in his last quoted paper. Thus, de Sitter's free space-time is defined by the line-element

$$ds^2 = \cos^2\frac{r}{R}\, c^2 dt^2 - dr^2 - R^2\sin^2\frac{r}{R}\,[d\phi^2 + \sin^2\phi d\theta^2] \qquad (116)$$

and is therefore, as we saw, a manifold of constant isotropic

*W. de Sitter, *Monthly Notices of R.A.S.*, November 1917.

curvature. Within matter Einstein's new equations, with $\lambda = 3/R^2$, are valid, *i.e.*,

$$G_{\iota\kappa} - \frac{3g_{\iota\kappa}}{R^2} = -\frac{8\pi}{c^2}\left(T_{\iota\kappa} - \tfrac{1}{2} g_{\iota\kappa} T\right). \qquad (120)$$

The isotropy of de Sitter's space-time, expressed by

$$G^{11} = G^{22} = G^{33} = G^{44} = \frac{3}{R^2},$$

as in (116^2), distinguishes it characteristically from Einstein's space-time. This goes hand in hand with $\rho_0 = 0$ outside of matter proper.

De Sitter's line-element differs from Einstein's by

$$g_{44} = \cos^2 \frac{r}{R}$$

instead of $g_{44} = 1$. Consequently, if the permanency of atoms be assumed as in Chapter V, the spectrum lines of distant stars should be displaced towards the red. If r be the distance of a star from an observer placed at the origin of coördinates, the observed wave-length should be increased from 1 to $1 : \cos \frac{r}{R}$, becoming infinite for $r = \frac{\pi}{2} R$, the greatest distance possible in a properly elliptic space. Manifestly, everything is at a standstill at the equatorial belt, *i.e.*, all along the polar of any observing station as pole. This, though sounding strangely, entails no actual difficulty at all. As to the spectrum shift of less distant celestial objects, de Sitter quotes the helium or B-stars which show a systematic displacement towards the red such as would correspond to a velocity of 4·5 km. per sec. If, as de Sitter suggests, one-third of this is considered as a gravitational Einstein-effect, the remainder may be accounted for by the decrease of g_{44}, and since the average distance of the B-stars is believed to be $r = 3.10^7$ astronomical units, we should have

$$1 - \cos \frac{3.10^7}{R} = 10^{-5},$$

and therefore a curvature radius $R = \frac{2}{3} 10^{10}$. But there is, for the present, nothing cogent in the attribution of the said

remainder of spectrum shift to the dwindling of g_{44} with mere distance, and it would certainly be premature either to reject or to accept the results of this attractive piece of reasoning.

Other consequences of the theory and a more thorough comparison with Einstein's solution will be found in de Sitter's paper. Here it will be enough to mention still that according to de Sitter's line-element the parallax of a star should reach a minimum at $r=\frac{1}{2}\pi R$, the greatest distance in the *polar* kind of space (which de Sitter prefers to the antipodal). This minimum, of the semi-parallax, is equal to $p=a/R$, if a be the distance of the earth from the sun. On the other hand, Einstein's line-element gives, for $r=\frac{1}{2}\pi R$, a vanishing parallax. Since de Sitter's minimum is at least as small as $p=10^{-10}=0''.00002$, one cannot reasonably hope to discriminate between the two proposals by direct observations of parallaxes, while indirect ones contain too many assumptions to be considered as crucial.

Soon after the publication of de Sitter's paper Einstein raised some objections to his form of the line-element. For these, however, not altogether crushing, the reader must be referred to Einstein's own paper (Berlin *Sitzungsberichte*, March 1918, pp. 270-272).

D. Gravitational Fields and Electrons.

The problem of the equilibrium of electricity constituting the electron as the structural element of matter proper, already attacked by G. Mie and others, has been taken up by Einstein in a paper of April 1919 (Berlin *Sitzungsber.*, pp. 349-356). The result of the investigation is that this tempting question cannot be completely answered by means of the field-equations alone. For details of the reasoning the reader must be referrred to the original paper. It will be enough to mention that *the fixed relation* between the universal constant λ in the amplified field-equations and the total mass of the universe, as related in Appendix **B**, is here given up. Space continues to be considered as closed but the curvature radius R and, therefore, the volume of the universe appears as independent of the total mass contained in it, though its macroscopic density ρ_0 is still treated as uniform.

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